# TRIANGULAR DE RHAM COHOMOLOGY OF COMPACT KÄHLER MANIFOLDS

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Abstract. We study the de Rham 1-cohomology $H^1_{DR}(M,G)$ of a smooth manifold $M$ with values in a Lie group $G$. By definition, this is the quotient of the set of flat connections in the trivial principal bundle $M\times G$ by the so-called gauge equivalence. We consider the case when $M$ is a compact Kähler manifold and $G$ is a solvable complex linear algebraic group of a special class which contains the Borel subgroups of all complex classical groups and, in particular, the group $T_n(\mathbb{C})$ of all triangular matrices. In this case, we get a description of the set $H^1_{DR}(M,G)$ in terms of the 1-cohomology of $M$ with values in the (abelian) sheaves of flat sections of certain flat Lie algebra bundles with fibre $\mathfrak{g}$ (the Lie algebra of $G$) or, equivalently, in terms of the harmonic forms on $M$ representing this cohomology.

## 0. Introduction

The paper is devoted to the study of the de Rham 1-cohomology $H^1_{DR}(M,G)$ of a smooth manifold $M$ with values in a Lie group $G$. If $G$ is non-abelian, then $H^1_{DR}(M,G)$ admits no natural group structure and is usually regarded as a set with a distinguished point. By definition, this is the quotient of the set of flat connections in the trivial principal bundle $M\times G$ by the so-called gauge equivalence, the distinguished point being the class of the zero connection. Note that the de Rham 1-cohomology set has two important interpretations. First, $H^1_{DR}(M,G)$ admits a natural injection into the Čech 1-cohomology set $H^1(M,G)$, the image being interpreted as the set of smoothly (or topologically) trivial flat principal bundles with base $M$ and structure group $G$. In the case $G=\mathbb{R}$, this correspondence is the classical de Rham isomorphism. Second, any flat connection determines the holonomy homomorphism $\pi_1(M)\to G$, giving rise to an injective mapping $H^1_{DR}(M,G)\to \operatorname{Hom}(\pi_1(M),G)/\operatorname{Int}G$. If $G$ is contractible (e.g., solvable and simply connected), then any smooth principal bundle with structure group $G$ is trivial, and both injections are bijections.

We consider the case when $M$ is a compact Kähler manifold and $G$ is a solvable complex linear algebraic group of a special class which contains the Borel subgroups

1991 *Mathematics Subject Classification.* Primary 53C07, 58A14, 32J27.

*Key words and phrases.* Non-abelian cochain complex, de Rham complex, Dolbeault complex, solvable algebraic group, Hodge property, flat connection, gauge transformation, flat fibre bundle, Hodge theory, harmonic form.

Work supported in part by the Russian Foundation for Fundamental Research (Grant 98-01-00329) (the second author).

of all complex classical groups and, in particular, the group $T_n(\mathbb{C})$ of all non-singular triangular $n\times n$-matrices. In this case, we get a description of the set $H^1_{DR}(M,G)$ in terms of the 1-cohomology of $M$ with values in the (abelian) sheaves of flat sections of certain flat Lie algebra bundles with fibre $\mathfrak{g}$ (the Lie algebra of $G$) or, equivalently, in terms of the harmonic forms on $M$ representing this cohomology. Our method is based on using the non-abelian Dolbeault cohomology set $H^{0,1}(M,G)$ of $M$ with values in $G$ that is defined similarly to the de Rham cohomology set and essentially exploits the properties of the Hodge decomposition on a compact Kähler manifold.

The paper is devided into two sections. The first one contains the necessary facts on non-abelian de Rham and Dolbeault cohomology. In particular, the technics of twisting described here is very important for the sequel. The proofs are omitted, referring the reader to [O3, O4]. We also expose some facts on Hodge theory for flat vector bundles.

Section 2 contains formulations and proofs of the main results of the paper. We introduce the so-called Hodge property for solvable complex algebraic groups. Lemma 2.2 gives a method to construct groups having this property, while Theorem 2.1 presents a list of such groups including the Borel subgroups of classical complex linear groups (one of these subgroups is $T_n(\mathbb{C})$). The Hodge property for $G$ implies, in particular, that the natural mapping $\Pi^*_{0,1} : H^1_{DR}(M,G) \to H^{0,1}(M,G)$ is surjective whenever $M$ is compact and Kähler. Theorem 2.2 describes the so-called canonical representatives of de Rham cohomology classes in terms of harmonic forms with values in certain Lie algebra bundles. The final result about $H^1_{DR}(M,G)$ is formulated as Theorem 2.3. In the last subsection, we formulate two results that can be proved by the same argument as Theorem 2.3. Theorem 2.4 gives a description of a twisted version of the de Rham cohomology of a compact Kähler manifold $M$ with values in the unipotent radical of a group $G$ having the Hodge property. In Theorem 2.5, the situation is considered when such a group $G$ is a subgroup of an algebraic group $\hat{G}$. Here we describe the quotient of the set of flat connections $\omega$ on $M\times\hat{G}$ such that $\Pi_{0,1}\omega$ takes its values in $\mathfrak{g}$ by the gauge equivalence determined by $G$.

In the case when $G = T_n(\mathbb{C})$, the main results of the paper were proved in the research thesis [Br1] of the first author. The thesis also contains applications of these results to the study of the fundamental group $\pi_1(M)$ of a compact Kähler manifold $M$ and, in particular, a classification of compact solvmanifolds that admit Kähler structures. These applications will be published in [Br2].

## 1. De Rham and Dolbeault cohomology with values in a Lie group

**1.1.** Here we discuss a non-linear cochain complex that coincides, in the classical abelian case, with the initial part of the usual de Rham complex of a smooth manifold. We follow [O3], Sections 4 and 5, and [O4], Section 2.

Let us first introduce some notation. Let $X$ be a topological space, $\mathcal{F}$ a sheaf of (non-necessary abelian) groups on $X$. Then one defines the 0-cohomology group $H^0(X,\mathcal{F})$ which coincides with the group of global sections $\Gamma(X,\mathcal{F})$. One also defines the Čech 1-cohomology $H^1(X,\mathcal{F})$ which in general is not a group, but merely a set with a distinguished point. Let $\mathfrak{U}$ be an open cover of $X$. In a usual way, one defines the groups of $p$-cochains $C^p(\mathfrak{U},\mathcal{F})$, $p\geq 0$, and the set of 1-cocycles

$$Z^1(\mathfrak{U},\mathcal{F}) = \{z\in C^1(\mathfrak{U},\mathcal{F}) \mid z_{ij}z_{jk} = z_{ik} \text{ in } U_i\cap U_j\cap U_k \neq \emptyset\}$$

There is an action $\rho$ of $C^0(\mathfrak{U},\mathcal{F})$ on $C^1(\mathfrak{U},\mathcal{F})$ given by

$$(\rho(a)(c))_{ij} = c_i z_{ij} c_j^{-1}.$$

The set $Z^1(\mathfrak{U},\mathcal{F})$ is invariant under $\rho$. Forming the quotient $H^1(\mathfrak{U},\mathcal{F}) = Z^1(\mathfrak{U},\mathcal{F})/\rho(C^0(\mathfrak{U},\mathcal{F}))$ and passing to a limit over all open covers $\mathfrak{U}$, one obtains the cohomology set $H^1(X,\mathcal{F})$. Its distinguished point $\varepsilon$ is the class of the unit cocycle $e \in Z^1(\mathfrak{U},\mathcal{F})$.

Let now $G$ be a Lie group. For any smooth manifold $M$, denote by $\mathcal{F}_G$ the sheaf of germs of smooth $G$-valued functions on $M$. In particular, $\mathcal{F} = \mathcal{F}_{\mathbb{R}}$ is the structure sheaf of $M$. Clearly, $\mathcal{F}_G$ is a sheaf of groups. The 0-cohomology $H^0(M,\mathcal{F}_G)$ is the group $F_G$ of global smooth functions $M \to G$, while the 1-cohomology set $H^1(M,\mathcal{F}_G)$ is usually identified with the set of all smooth principal bundles with base $M$ and structure group $G$ (regarded up to isomorphy). Namely, if $\mathfrak{U} = (U_i)$ is an open cover of $M$, then any $z \in Z^1(\mathfrak{U},\mathcal{F}_G)$ determines the principal bundle with the transition functions $z_{ij}$ got from $M \times G$ by twisting with the help of $z$; this is the bundle corresponding to the cohomology class $\zeta \in H^1(M,\mathcal{F}_G)$ of the cocycle $z$. The unit element $\varepsilon \in H^1(M,\mathcal{F}_G)$ corresponds to the trivial bundle $M \times G$.

Clearly, we may identify the constant sheaf $G$ on $M$ with the subsheaf of $\mathcal{F}_G$ consisting of germs of *flat* (i.e., locally constant) functions. Now, $H^0(M,G)$ is the group of all flat functions $M \to G$. The set $H^1(M,G)$ will be interpreted as the set of *flat* principal bundles with base $M$ and structure group $G$, i.e., principal bundles with locally constant transition functions, regarded up to corresponding isomorphy.

Let $\mathfrak{g}$ be the tangent Lie algebra of $G$ and $\Phi_{\mathfrak{g}} = \bigoplus_{p\geq 0} \Phi_{\mathfrak{g}}^p$ the graded sheaf of $\mathfrak{g}$-valued smooth forms on $M$. This is a sheaf of graded Lie superalgebras, the bracket $[\,,\,]$ being induced by that of $\mathfrak{g}$. Denoting $A_{\mathfrak{g}}^p = \Gamma(G,\Phi_{\mathfrak{g}}^p)$, we obtain the usual $\mathfrak{g}$-valued de Rham complex $(A_{\mathfrak{g}},d)$, where $A_{\mathfrak{g}} = \bigoplus_{p\geq 0} A_{\mathfrak{g}}^p$ has a structure of graded Lie superalgebra and the exterior derivative $d$ is its derivation of degree 1.

In order to define the desired non-linear complex, we denote by $\varpi \in A_{\mathfrak{g}}^1$ the canonical 1-form on $G$ assigning to any tangent vector $v$ at $g \in G$ the vector $d_g r_g^{-1}(v) \in \mathfrak{g}$, where $r_g : x \mapsto xg$ is the right translation corresponding to $g$. Consider the triple of groups $R_G = \{R_G^0, R_G^1, R_G^2\} = \{F_G, A_{\mathfrak{g}}^1, A_{\mathfrak{g}}^2\}$ and define the coboundary operators $\delta_0 : R_G^0 \to R_G^1$ and $\delta_1 : R_G^1 \to R_G^2$ by

$$\delta_0(g) = g^*(\varpi), \quad g \in F_G, \tag{1.1}$$

$$\delta_1(\alpha) = d\alpha - \frac{1}{2}[\alpha,\alpha], \quad \alpha \in A_{\mathfrak{g}}^1. \tag{1.2}$$

Note that $R_G^0 = F_G$ acts on the graded algebra $A_{\mathfrak{g}}$ by automorphisms induced by the adjoint representation $\mathrm{Ad}$ of $G$. Denoting this action by $\mathrm{Ad}$, we have

$$d(\mathrm{Ad}\, g(\alpha)) = \mathrm{Ad}\, g(d\alpha) + [\delta_0(g), \mathrm{Ad}\, g(\alpha)]. \tag{1.3}$$

One checks easily that $\delta_0 : F_G \to A_{\mathfrak{g}}^1$ is a *crossed homomorphism* with respect to $\mathrm{Ad}$, i.e.,

$$\delta_0(gh) = \delta_0(g) + \mathrm{Ad}\, g(\delta_0(h)). \tag{1.4}$$

This gives rise to the following affine action of $R_G^0$ on $R_G^1 = A_{\mathfrak{g}}^1$ by so-called *gauge transformations*:

$$\rho(g)(\alpha) = \mathrm{Ad}\, g(\alpha) + \delta_0(g) \tag{1.5}$$

In what follows, $G$ often will be a Lie subgroup of the linear group $\mathrm{GL}_m(\mathbb{R})$. In this case, the coboundary operators $\delta_p$ and the actions Ad and $\rho$ can be written as follows:

$$\begin{aligned}\delta_0(g) &= (dg)g^{-1},\\ \delta_1(\alpha) &= d\alpha - \alpha\wedge\alpha,\\ \operatorname{Ad} g(\alpha) &= g\alpha g^{-1},\\ \rho(g)(\alpha) &= g\alpha g^{-1} + (dg)g^{-1}.\end{aligned}$$

Using (1.3), one easily verifies the following relation:

$$\delta_1\circ\rho(g) = \operatorname{Ad} g\circ\delta_1. \tag{1.6}$$

This means that the triple $R_G$ together with the mappings $\delta_p$, $p = 0, 1$, and the actions Ad and $\rho$ is a *non-abelian cochain complex* in the sense of [O3, O4]. It is called the *de Rham complex with values in* $G$.

Introduce the sets of cocycles

$$Z^p(R_G) = \operatorname{Ker}\delta_p = \delta_p^{-1}(0),\quad p = 0, 1.$$

Then $Z^0(R_G)$ is a subgroup of $F_G$. Now, the subset $Z^1(R_G)$ is invariant under $\rho$ due to (1.6). We define the *de Rham cohomology of $M$ with values in $G$* (or the cohomology of the complex $R_G$) by

$$\begin{aligned}H^0_{DR}(M,G) &= H^0(R_G) = Z^0(R_G),\\ H^1_{DR}(M,G) &= H^1(R_G) = Z^1(R_G)/\rho(F_G).\end{aligned}$$

Then $H^0_{DR}(M,G) = H^0(M,G)$. If $G$ is non-abelian, then the set $H^1_{DR}(M,G)$ does not admit any natural group structure. We regard it as a set with the distinguished point $\varepsilon = \rho(F_G)(0)$. The cohomology class of a cocycle $\omega\in Z^1(R_G)$ will be denoted by $[\omega]$.

Note that on the sheaf level we get the following exact sequence of sheaves:

$$e\to G\xrightarrow{i}\mathcal{F}_G\xrightarrow{\delta_0}\Phi^1_{\mathfrak{g}}\xrightarrow{\delta_1}\Phi^2_{\mathfrak{g}}, \tag{1.7}$$

where $i$ is the natural injection. It implies the following relation between the de Rham and the Čech 1-cohomology (see [O3], Section 5).

**Proposition 1.1.** *We have the following exact sequence of sets with distinguished points:*

$$e\to H^1_{DR}(M,G)\xrightarrow{\mu} H^1(M,G)\xrightarrow{i_1^*} H^1(M,\mathcal{F}_G). \tag{1.8}$$

*Here $i_1^*$ is determined by $i$, while $\mu$ is defined as follows. For any cocycle $\omega\in Z^1(R_G)$, choose an open cover $\mathfrak{U} = (U_i)$ of $M$ such that $\omega = \delta_0(c_i)$ in any $U_i$ for certain smooth $c_i : U_i\to G$. Then $z_{ij} = c_i^{-1}c_j$ form a cocycle $z = (z_{ij})\in Z^1(\mathfrak{U},G)$, and $\mu$ sends $[\omega]$ to the cohomology class of $z$.*

The exactitude of (1.8) means that $\mu$ maps bijectively the set $H^1_{DR}(M,G)$ onto the subset of those flat bundles from $H^1(M,G)$ which are trivial as smooth bundles.

In the case $G = \mathbb{R}$, the complex $R_G$ is, clearly, a part of the classical de Rham complex, and (1.8) gives the de Rham theorem for the 1-cohomology.

If $G$ is non-abelian, then (1.8) is not an exact sequence of groups and their homomorphisms, and so the fibres of the mapping $i_1^*$ cannot be described in terms of its kernel. To describe these fibres, it is necessary to consider twisted versions of the de Rham complex.

Now we discuss certain twisting constructions. Let $\mathcal{S}$ be a sheaf of sets on $X$ and let us denote by $\mathcal{A}ut\,\mathcal{S}$ the sheaf of germs of automorphisms of this sheaf. Its sections over an open set $U \subset X$ are, by definition, bijective continuous mappings $\mathcal{S}|U \to \mathcal{S}|U$ leaving each stalk invariant. Clearly, $\mathcal{A}ut\,\mathcal{S}$ is a sheaf of groups. Let $\mathfrak{U} = (U_i)$ be an open cover of $X$, and fix a cocycle $z \in Z^1(\mathfrak{U}, \mathcal{A}ut\,\mathcal{S})$. Then we can twist $\mathcal{S}$ with the help of $z$ getting a new sheaf $\mathcal{S}^z$ on $X$. This means that we glue together any two sheaves $\mathcal{S}|U_i$ and $\mathcal{S}|U_j$ over $U_i \cap U_j \neq \emptyset$ identifying $a_i \in (\mathcal{S}|U_i)_x$ and $a_j \in (\mathcal{S}|U_j)_x$, $x \in U_i \cap U_j$, under the condition $a_i = z_{ij}(a_j)$. If $S$ is a sheaf of groups and we use automorphisms of a sheaf of groups, then, clearly, $\mathcal{S}^z$ is a sheaf of groups, too, etc.

The *twisted sheaf* $\mathcal{S}^z$ depends, up to isomorphism, only on the cohomology class $\zeta \in H^1(X, \mathcal{A}ut\,\mathcal{S})$ determined by $z$. E.g., if $z$ is cohomologous to $e$, i.e., $z_{ij} = c_i c_j^{-1}$ for a 0-cochain $c \in C^0(\mathfrak{U}, \mathcal{A}ut\,\mathcal{S})$, then $a_i = z_{ij}(a_j)$ implies $c_i^{-1}(a_i) = c_j^{-1}(a_j)$. Thus, the correspondence $(a_i) \mapsto (c_i^{-1}(a_i))$ is an isomorphism of $\mathcal{S}^z$ onto $\mathcal{S}$.

Note that any section $s \in \Gamma(U, \mathcal{S}^z)$ is given by a 0-cochain $(s_i)$, where $s_i \in \Gamma(U \cap U_i, \mathcal{S})$, satisfying $s_i = z_{ij}(s_j)$ over $U \cap U_i \cap U_j$.

To twist the de Rham complex with values in a Lie group $G$, suppose that a cocycle $\mathfrak{z} \in Z^1(\mathfrak{U}, \operatorname{Aut} G)$ is given, $\operatorname{Aut} G$ being the group of automorphisms of $G$. Consider the twisted sheaf of groups $G^{\mathfrak{z}}$. We can realize it as the sheaf $\mathcal{C}_E$ of locally constant sections of the flat group bundle $E$ which is got from $M \times G$ by twisting with the help of $\mathfrak{z}$. Any automorphism of $G$ induces an automorphism of $\mathcal{F}_G$, and hence we get the twisted sheaf $(\mathcal{F}_G)^{\mathfrak{z}}$ which is the sheaf $\mathcal{F}_E$ of smooth sections of $E$. We also have the cocycle $d\mathfrak{z}$ of automorphisms of the Lie algebra $\mathfrak{g}$ giving rise to the twisted sheaves $(\Phi^p_{\mathfrak{g}})^{d\mathfrak{z}}$, $p \geq 0$; these are the sheaves $\Phi^p_{\mathfrak{e}}$ of $p$-forms with values in the Lie algebra bundle $\mathfrak{e}$ which is got from $M \times \mathfrak{g}$ by twisting with the help of $d\mathfrak{z}$.

Define now the graded Lie superalgebra $A_{\mathfrak{e}} = \bigoplus_{p \geq 0} A^p_{\mathfrak{e}}$, where $A^p_{\mathfrak{e}} = \Gamma(M, \Phi^p_{\mathfrak{e}})$. Setting

$$(d\alpha)_i = d\alpha_i,$$

we correctly define a derivation $d$ of $A_{\mathfrak{e}}$ giving rise to the twisted de Rham complex $(A_{\mathfrak{e}}, d)$. Its cohomology is denoted by $H^p_{DR}(M, \mathfrak{e})$, by a generalized de Rham theorem it is isomorphic to $H^p(M, \mathcal{C}_{\mathfrak{e}})$. The non-linear twisted complex is defined as the triple $R_E = \{R^0_E, R^1_E, R^2_E\}$, where $R^0_E = F_E = \Gamma(M, \mathcal{F}_E)$, $R^p_E = A^p_{\mathfrak{e}}$, $p = 1, 2$. The coboundary operators and the actions of $F_E$ are correctly defined by

$$
\begin{aligned}
(\delta_0(f))_i &= (\delta_0(f_i)) \text{ for } f = (f_i) \in F_E,\\
(\delta_1(\alpha))_i &= (\delta_1(\alpha_i)) \text{ for } \alpha = (\alpha_i) \in A^1_{\mathfrak{e}},\\
(\operatorname{Ad} f(\alpha))_i &= (\operatorname{Ad} f_i(\alpha)_i) \text{ for } \alpha = (\alpha_i) \in A^p_{\mathfrak{e}},\\
(\rho(f)(\alpha))_i &= (\rho(f_i)(\alpha)_i) \text{ for } \alpha = (\alpha_i) \in A^1_{\mathfrak{e}}.
\end{aligned}
\tag{1.9}
$$

As above, we obtain a non-abelian cochain complex. One defines the 0-cohomology group $H^0_{DR}(M, E) = H^0(R_E) = Z^0(R_E)$ and the 1-cohomology set $H^1_{DR}(M, E)$

$H^1(R_E) = Z^1(R_E)/\rho(F_E)$. The group $H^0_{DR}(M, E)$ coincides with the group $H^0(M, \mathcal{C}_E)$ of flat sections of $E$. Generalizing (1.8), we get the following exact sequence of sets with distinguished points:

$$e \to H^1_{DR}(M, E) \xrightarrow{\mu} H^1(M, \mathcal{C}_E) \xrightarrow{i_1^*} H^1(M, \mathcal{F}_G). \tag{1.10}$$

Let us denote by $\operatorname{Int} h$ the inner automorphism $x \mapsto hxh^{-1}$ of a group $H$ determined by $h \in H$. Suppose that $\mathfrak{z} = \operatorname{Int} z$, i.e., $\mathfrak{z}_{ij} = \operatorname{Int} z_{ij}$, where $z = (z_{ij}) \in Z^1(\mathfrak{U}, G)$. Denote by $P$ the flat principal bundle with the structure group $G$ over $M$ corresponding to $z$. Then $E = \operatorname{Int} P$ is the group bundle associated to $P$ by the action Int of $G$ on itself by inner automorphisms, and $\mathfrak{e} = \operatorname{Ad} P$ is the Lie algebra bundle associated to $P$ by the adjoint representation Ad of $G$ on $\mathfrak{g}$. The vector space $R^1_{\operatorname{Int} P} = A^1_{\operatorname{Ad} P}$ can be interpreted as the space of connections in $P$ (regarded as a smooth principal bundle), while $\delta_1(\alpha) \in R^2_{\operatorname{Int} P} = A^2_{\operatorname{Ad} P}$ is interpreted as the curvature of a connection $\alpha$, so that $Z^1(R_{\operatorname{Int} P})$ is the set of connections with zero curvature (or *flat* connections). Now, $R^0_{\operatorname{Int} P} = F_{\operatorname{Int} P}$ is identified with the group of smooth automorphisms of the principal bundle $P$ (inducing the identity mapping of the base $M$), and $\rho$ is the action of this group on $A^1_{\operatorname{Ad} P}$ by gauge transformations. By definition, $H^1(R_{\operatorname{Int} P}) = H^1_{DR}(M, \operatorname{Int} P)$ is the quotient of $Z^1(R_{\operatorname{Int} P})$ by this action. Also $H^0(R_{\operatorname{Int} P}) = H^0(M, \mathcal{C}_{\operatorname{Int} P})$ is the group of flat automorphisms of $P$. There exists a natural bijection of $H^1(R_{\operatorname{Int} P})$ onto the subset $(i_1^*)^{-1}(i_1^*(\zeta)) \subset H^1(M, G)$ of all flat principal bundles isomorphic to $P$ as smooth bundles (or, which is the same, onto the set of all flat structures in the bundle $P$ regarded as a smooth principal bundle, up to automorphisms of this smooth bundle), see [O3], Section 5.

Now we consider the special case when $P$ is trivial (as a smooth bundle). This means that the cohomology class $\zeta$ of $z$ lies in $\operatorname{Im} \mu$ (see (1.8)). More precisely, we may suppose that $z_{ij} = c_i^{-1} c_j$, where $c_i \in \Gamma(U_i, \mathcal{F}_G)$. Then $\gamma = \delta_0(c_i)$ is a well-defined global form from $Z^1(R_G)$ such that $[\gamma]$ satisfies $\mu([\gamma]) = \zeta$. The cochain $(c_i)$ determines an isomorphism of groups $t : R^0_{\operatorname{Int} P} \to R^0_G = F_G$ given by

$$t((g_i)) = c_i g_i c_i^{-1} \ \text{ in } \ U_i. \tag{1.11}$$

Similarly, we get an isomorphism of graded Lie superalgebras $\tau : A_{\operatorname{Ad} P} \to A_{\mathfrak{g}}$ given on the $p$-components by

$$\tau_p((\alpha_i)) = \operatorname{Ad} c_i(\alpha_i) \ \text{ in } \ U_i, \quad (\alpha_i) \in A^p_{\operatorname{Ad} P}. \tag{1.12}$$

This is not an isomorphism of complexes. More precisely, we see from (1.3) that

$$\tau \circ d = (d - \operatorname{ad} \gamma) \circ \tau,$$

where $\operatorname{ad} \gamma : \alpha \mapsto [\gamma, \alpha]$ is the adjoint operator in $A_{\mathfrak{g}}$ associated with $\gamma$. Thus, the following is true:

**Proposition 1.2.** *The mapping $\tau$ defined by (1.12) is an isomorphism of complexes $(A_{\operatorname{Ad} P}, d) \to (A_{\mathfrak{g}}, d - \operatorname{ad} \gamma)$.*

Now we establish a relation between the 1-cohomology sets of $R_G$ and $R_{\operatorname{Int} P}$ (in the case when $P$ is trivial as a smooth bundle). Define $r_\gamma : A^1_{\operatorname{Int} P} \to A^1_{\mathfrak{g}}$ by

$$r_\gamma((\alpha_i)) = \tau_1(\alpha_i) + \gamma = \operatorname{Ad} c_i(\alpha_i) + \gamma = \rho(c_i)(\alpha_i). \tag{1.13}$$

Clearly, this is a well-defined bijective affine mapping.

**Proposition 1.3.** *In the notation given by (1.11), (1.12), (1.13), the following relations hold:*

$$\delta_1 \circ r_\gamma = \tau_2 \circ \delta_1,$$
$$r_\gamma \circ \rho(g) = \rho(t(g)) \circ r_\gamma, \quad g \in F_{\mathrm{Int}\,P}.$$

*Thus, $r_\gamma$ maps $Z^1(R_{\mathrm{Int}\,P})$ onto $Z^1(R_G)$ and induces a bijection $r_\gamma^* : H^1_{DR}(M, \mathrm{Int}\,P) \to H^1_{DR}(M, G)$ taking $\varepsilon$ to $[\gamma]$.*

The proof is straightforward. The statement is also implied by Proposition 1.5 of [O4].

The construction of the de Rham complex gives a covariant functor $G \mapsto R_G$ from the category of Lie groups into that of cochain complexes (see [O4]). In fact, to any smooth homomorphism of Lie groups $f : G \to Q$ there corresponds the triple $\{f_0, f_1, f_2\}$ of homomorphisms $f_p : R^p_G \to R^p_Q$, where $f_0(g) = f \circ g,\ g \in R^0_G = F_G$, and $f_p(\alpha) = df \circ \alpha,\ \alpha \in R^p_G = A^p_{\mathfrak g},\ p = 1, 2$, satisfying

$$\delta_p \circ f_p = f_{p+1} \circ \delta_p,\ p = 0, 1,$$
$$f_1 \circ \rho(g) = \rho(f_0(g)) f_1.$$

Clearly, $f$ determines a homomorphism of groups $f_0^* : H^0_{DR}(M, G) \to H^0_{DR}(M, Q)$ and a homomorphism of sets with distinguished points $f_1^* : H^1_{DR}(M, G) \to H^1_{DR}(M, Q)$.

**1.2.** Here we consider a theory, analogous to that exposed in n°1.1, in which flat bundles and locally constant sections or functions are replaced by holomorphic ones. The abelian model is the classical Dolbeault complex of a complex analytic manifold. We follow [O3], Section 6.

Let $G$ be a complex Lie group. For any complex manifold $M$, denote by $\mathcal{O}_G$ the sheaf of germs of $G$-valued holomorphic functions on $M$. In particular, $\mathcal{O} = \mathcal{O}_{\mathbb{C}}$ is the structure sheaf of $M$. Clearly, $H^0(M, \mathcal{O}_G)$ is the group of all $G$-valued holomorphic functions on $M$, while $H^1(M, \mathcal{O}_G)$ can be interpreted as the set of all principal holomorphic bundles with the base $M$ and the structure group $G$, considered up to holomorphic isomorphisms leaving any point of $M$ fixed. The unit cohomology class $\varepsilon$ corresponds to the trivial bundle $M \times G$. As in Section 1.1, we will denote by $\mathcal{F}_G$ the sheaf of smooth $G$-valued functions on $M$ and by $\Phi_{\mathfrak g}$ the sheaf of $\mathfrak g$-valued smooth forms on $M$. This is a sheaf of bigraded complex Lie superalgebras, the bigrading being defined by $\Phi_{\mathfrak g} = \bigoplus_{p,q \geq 0} \Phi^{p,q}_{\mathfrak g}$, where $\Phi^{p,q}_{\mathfrak g}$ is the sheaf of forms of type $(p, q)$. Let $\Pi_{p,q} : \Phi^{p+q} \to \Phi^{p,q}$ be the natural projection. As usually, denote

$$\partial\alpha = \Pi_{p+1,q} d\alpha,\ \bar\partial\alpha = \Pi_{p,q+1} d\alpha, \quad \alpha \in \Phi_{p,q},$$

Then

$$d = \partial + \bar\partial, \quad \partial^2 = \bar\partial^2 = [\partial, \bar\partial] = 0.$$

Denote $A^{p,q}_{\mathfrak g} = \Gamma(M, \Phi^{p,q}_{\mathfrak g})$. Then $A_{\mathfrak g} = \bigoplus_{p,q \geq 0} A^{p,q}_{\mathfrak g}$ is a bigraded complex Lie superalgebra, and $\partial$ and $\bar\partial$ are derivations of bidegree $(1, 0)$ and $(0, 1)$, respectively. We get the bigraded Dolbeault complexes $(A_{\mathfrak g}, \bar\partial)$ and $(A_{\mathfrak g}, \partial)$. Denote by

$$\Omega^p_{\mathfrak g} = \{\alpha \in \Phi^{p,0}_{\mathfrak g} \mid \bar\partial\alpha = 0\},$$
$$\overline{\Omega}^p_{\mathfrak g} = \{\alpha \in \Phi^{0,p}_{\mathfrak g} \mid \partial\alpha = 0\},$$

the subsheaves of holomorphic and antiholomorphic $p$-forms, respectively. We have the classical Dolbeault isomorphisms

$$H^{p,q}(M,\mathfrak{g}) \stackrel{\mathrm{def}}{=} H^q(A^{p,*}_{\mathfrak{g}},\bar\partial) \simeq H^q(M,\Omega^p_{\mathfrak{g}}), \tag{1.14}$$

$$\overline{H}^{p,q}(M,\mathfrak{g}) \stackrel{\mathrm{def}}{=} H^p(A^{*,q}_{\mathfrak{g}},\partial) \simeq H^p(M,\overline{\Omega}^q_{\mathfrak{g}}). \tag{1.15}$$

Now we define the non-linear Dolbeault complex with values in $G$. Consider the triple of groups $\overline{R}_G = \{\overline{R}^0_G, \overline{R}^1_G, \overline{R}^2_G\} = \{F_G, A^{0,1}_{\mathfrak{g}}, A^{0,2}_{\mathfrak{g}}\}$ and define the coboundary operators $\bar\delta_0 : \overline{R}^0_G \to \overline{R}^1_G$, $\bar\delta_1 : \overline{R}^1_G \to \overline{R}^2_G$ and the action $\bar\rho$ of $\overline{R}^0_G$ on $\overline{R}^1_G$ by

$$\begin{aligned}
\bar\delta_0(g) &= \Pi_{0,1} g^*(\varpi),\\
\bar\delta_1(\alpha) &= \bar\partial\alpha - \frac{1}{2}[\alpha,\alpha],\\
\bar\rho(g)(\alpha) &= \mathrm{Ad}\, g(\alpha) + \bar\delta_0(g).
\end{aligned}$$

Then we get a non-abelian cochain complex in the sense of [O3, O4], called the *Dolbeault complex with values in* $G$. If $G$ is a Lie subgroup of $\mathrm{GL}_m(\mathbb{C})$, then

$$\begin{aligned}
\bar\delta_0(g) &= (\bar\partial g)g^{-1},\\
\bar\delta_1(\alpha) &= \bar\partial\alpha - \alpha\wedge\alpha,\\
\bar\rho(g)(\alpha) &= g\alpha g^{-1} + (\bar\partial g)g^{-1}.
\end{aligned}$$

The cocycles of this complex are defined by

$$Z^p(\overline{R}_G) = \mathrm{Ker}\,\bar\delta_p = \bar\delta_p^{-1}(0), \quad p = 0,1.$$

Then $Z^0(\overline{R}_G)$ is a subgroup of $F_G$. Now, the subset $Z^1(\overline{R}_G)$ is invariant under $\bar\rho$. We define the *Dolbeault cohomology of* $M$ *with values in* $G$ (or the cohomology of $\overline{R}_G$) by

$$\begin{aligned}
H^0(\overline{R}_G) &= Z^0(\overline{R}_G),\\
H^{0,1}(M,G) = H^1(\overline{R}_G) &= Z^1(\overline{R}_G)/\bar\rho(F_G).
\end{aligned}$$

Note that on the sheaf level we get the following exact sequence of sheaves:

$$e \to \mathcal{O}_G \xrightarrow{i} \mathcal{F}_G \xrightarrow{\bar\delta_0} \Phi^{0,1}_{\mathfrak{g}} \xrightarrow{\bar\delta_1} \Phi^{0,2}_{\mathfrak{g}},$$

where $i$ is the natural injection. It implies the following relation between the Dolbeault and the Čech 1-cohomology (see [O3], Section 6).

**Proposition 1.4.** *We have the following exact sequence of sets with distinguished points:*

$$e \to H^{0,1}(M,G) \xrightarrow{\bar\mu} H^1(M,\mathcal{O}_G) \xrightarrow{i_1^*} H^1(M,\mathcal{F}_G).$$

*Here* $i_1^*$ *is determined by* $i$, *while* $\bar\mu$ *is defined as follows. For any cocycle* $\omega \in Z^1(\overline{R}_G)$, *choose an open cover* $\mathfrak{U} = (U_i)$ *of* $M$ *such that* $\omega = \bar\delta_0(c_i)$ *in any* $U_i$ *for certain smooth* $c_i : U_i \to G$. *Then* $z_{ij} = c_i^{-1}c_j$ *determine a cocycle* $z = (z_{ij}) \in$

*$Z^1(\mathfrak{U}, \mathcal{O}_G)$, and $\bar\mu$ takes the Dolbeault cohomology class $[\omega]$ of $\omega$ to the cohomology class of $z$.*

This means that $\bar\mu$ maps bijectively the set $H^{0,1}(M, G)$ onto the subset of those holomorphic bundles from $H^1(M, \mathcal{O}_G)$ which are trivial as smooth bundles.

As in Section 1.1, there exist twisted versions of the Dolbeault complex. Denote by $\operatorname{Aut}_h G$ the group of holomorphic automorphisms of $G$ and choose a cocycle $\mathfrak{z} \in Z^1(\mathfrak{U}, \mathcal{F}_{\operatorname{Aut}_h G})$. Clearly, $\mathcal{O}_{\operatorname{Aut}_h G}$ acts on $\mathcal{O}_G$ and $\mathcal{F}_G$. We get the holomorphic group bundle $E$ obtained by twisting $M \times G$ with the help of $\mathfrak{z}$ and the sheaves $\mathcal{O}_E = (\mathcal{O}_G)^{\mathfrak{z}}$ and $\mathcal{F}_E = (\mathcal{F}_G)^{\mathfrak{z}}$ of holomorphic and smooth sections of $E$, respectively. The cocycle $d\mathfrak{z}$ gives rise to the twisted sheaves $(\Phi^{p,q}_{\mathfrak{g}})^{d\mathfrak{z}}$, $p, q \geq 0$; these are the sheaves $\Phi^{p,q}_{\mathfrak{e}}$ of $(p,q)$-forms with values in the holomorphic Lie algebra bundle $\mathfrak{e}$ which is got from $M \times \mathfrak{g}$ by twisting with the help of $d\mathfrak{z}$. Clearly, the operator $\bar\partial$ is well-defined in $\Phi_{\mathfrak{e}} = \bigoplus_{p,q\geq 0} \Phi^{p,q}_{\mathfrak{e}}$, and one can define the subsheaf of holomorphic $p$-forms $\Omega^p_{\mathfrak{e}} \subset \Phi^{p,0}_{\mathfrak{e}}$. Now, one can consider the bigraded Dolbeault complex $(A_{\mathfrak{g}}, \bar\partial)$, where $A_{\mathfrak{g}} = \Gamma(M, \Phi_{\mathfrak{e}})$. Then there are the Dolbeault isomorphisms similar to (1.14):

$$H^{p,q}(M, \mathfrak{e}) \stackrel{\text{def}}{=} H^q(A^{p,*}_{\mathfrak{e}}, \bar\partial) \simeq H^q(M, \Omega^p_{\mathfrak{e}}). \tag{1.16}$$

The twisted version of the Dolbeault complex $\overline{R}_G$ is $\overline{R}_E = \{\overline{R}^0_E, \overline{R}^1_E, \overline{R}^2_E\}$, where $\overline{R}^0_E = F_E$, $\overline{R}^p_E = A^{0,p}_{\mathfrak{e}}$, $p = 1, 2$. The coboundary operators and the actions of $F_E$ are correctly defined by formulas similar to (1.9). The 0-cohomology of this cochain complex is the group $\Gamma(M, \mathcal{O}_E)$ of all holomorphic sections of $E$; the 1-cohomology will be denoted $H^{0,1}(M, E)$. Let us consider the case when $\mathfrak{z} = \operatorname{Int} z$, where $z = (z_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O}_G)$. Denote by $P$ the principal holomorphic bundle over $M$ corresponding to $z$. Then $E = \operatorname{Int} P$ is the group bundle associated to $P$ by the action Int of $G$ on itself by inner automorphisms, and $\mathfrak{e} = \operatorname{Ad} P$ is the Lie algebra bundle associated to $P$ by the adjoint representation Ad of $G$ on $\mathfrak{g}$.

Now we consider the special case when $P$ is trivial as a smooth bundle. We may suppose that $z_{ij} = c_i^{-1} c_j$, where $c_i \in \Gamma(U_i, \mathcal{F}_G)$. Then $\gamma = \bar\delta_0(c_i)$ is a well-defined global form from $Z^1(\bar R_G)$, and its cohomology class $[\gamma]$ satisfies $\bar\mu([\gamma]) = \zeta$, where $\zeta$ is the cohomology class of $z$. Precisely as in Section 1.1, we get an isomorphism of groups $t : F_{\operatorname{Int} P} \to F_G$ given by (1.11) and an isomorphism of bigraded Lie superalgebras $\tau : A_{\operatorname{Ad} P} \to A_{\mathfrak{g}}$ given by (1.12). Define a mapping $\bar r_\gamma : A^{0,1}_{\operatorname{Int} P} \to A^{0,1}_{\mathfrak{g}}$ by

$$\bar r_\gamma((\alpha_i)) = \tau_1(\alpha_i) + \gamma = \operatorname{Ad} c_i(\alpha_i) + \gamma = \bar\rho(c_i)(\alpha_i). \tag{1.17}$$

The following statements are easily verified.

**Proposition 1.5.** *The mapping $\tau$ is an isomorphism of complexes $(A_{\operatorname{Ad} P}, \bar\partial) \to (A_{\mathfrak{g}}, \bar\partial - \operatorname{ad} \gamma)$.*

**Proposition 1.6.** *The following relations hold:*

$$\bar\delta_1 \circ \bar r_\gamma = \tau_2 \circ \bar\delta_1,$$

$$\bar r_\gamma \circ \bar\rho(g) = \bar\rho(t(g)) \circ \bar r_\gamma, \quad g \in \overline{R}^0_{\operatorname{Int} P}.$$

*Thus, $\bar r_\gamma$ maps $Z^1(\overline{R}_{\operatorname{Int} P})$ onto $Z^1(\overline{R}_G)$ and induces a bijection $\bar r^*_\gamma : H^{0,1}(M, \operatorname{Int} P) \to H^{0,1}(M, G)$ taking $\varepsilon$ to $[\gamma]$.*

In what follows, we will consider the case when $E$ is flat, i.e., is obtained by twisting $M \times G$ with the help of a cocycle $\mathfrak{z} \in Z^1(\mathfrak{U}, \operatorname{Aut}_h G)$. Then the operator

$\partial$ is also well-defined in $\Phi_{\mathfrak{e}}$, and one can define the subsheaf of antiholomorphic $q$-forms $\overline{\Omega}^q_{\mathfrak{e}} \subset \Phi^{0,q}_{\mathfrak{e}}$. We also get the bigraded complex $(A_{\mathfrak{g}}, \partial)$ and the Dolbeault isomorphisms similar to (1.15):

$$\overline{H}^{p,q}(M,\mathfrak{e}) \stackrel{\text{def}}{=} H^p(A^{*,q}_{\mathfrak{e}}, \partial) \simeq H^p(M, \overline{\Omega}^q_{\mathfrak{e}}). \tag{1.18}$$

As in the case of the de Rham complex, the correspondence $G \mapsto \overline{R}_G$ is a covariant functor from the category of complex Lie groups into that of cochain complexes. In particular, any holomorphic homomorphism of complex Lie groups $f : G \to Q$ determines a homomorphism of sets with distinguished points $f_1^* : H^{0,1}(M,G) \to H^{0,1}(M,Q)$.

**1.3**. There is an important relationship between the de Rham and the Dolbeault complexes with values in the same complex Lie group $G$. It corresponds to the natural inclusion $\iota_G : G \to \mathcal{O}_G$. One checks easily that the triple $\{\mathrm{id}, \Pi_{0,1}, \Pi_{0,2}\}$ is a homomorphism of complexes $R_G \to \overline{R}_G$, i.e.,

$$\bar{\delta}_1 \circ \Pi_{0,1} = \Pi_{0,2} \circ \delta_1,$$
$$\bar{\rho}(g) \circ \Pi_{0,1} = \Pi_{0,1} \circ \rho(g), \quad g \in F_G.$$

Hence we get the following commutative diagram:

$$\begin{tikzcd} H^1_{DR}(M,G) \arrow[r, "\mu"] \arrow[d, "\Pi^*_{0,1}"'] & H^1(M,G) \arrow[r, "i^*"] \arrow[d, "\iota^*_G"] & H^1(M,\mathcal{F}_G) \arrow[d, equal] \\ H^{0,1}(M,G) \arrow[r, "\bar{\mu}"'] & H^1(M,\mathcal{O}_G) \arrow[r, "i^*"'] & H^1(M,\mathcal{F}_G), \end{tikzcd} \tag{1.19}$$

where $\Pi^*_{0,1}$ is induced by $\Pi_{0,1}$.

More generally, suppose that we have a flat group bundle $E$ corresponding to a cocycle $\mathfrak{z} \in Z^1(\mathfrak{U}, \mathrm{Aut}_h\, G)$. Regarding $E$ as a holomorphic Lie group bundle, we get the commutative diagram

$$\begin{tikzcd} H^1_{DR}(M,E) \arrow[r, "\mu"] \arrow[d, "\Pi^*_{0,1}"'] & H^1(M,\mathcal{C}_E) \arrow[r, "i^*"] \arrow[d, "\iota^*_E"] & H^1(M,\mathcal{E}) \arrow[d, equal] \\ H^{0,1}(M,E) \arrow[r, "\bar{\mu}"'] & H^1(M,\mathcal{O}_E) \arrow[r, "i^*"'] & H^1(M,\mathcal{E}), \end{tikzcd} \tag{1.20}$$

where $\Pi^*_{0,1}$ is induced by $\Pi_{0,1}$ and $\iota^*_E$ by the natural inclusion $\iota_E : \mathcal{C}_E \to \mathcal{O}_E$.

We also note that for any $\gamma \in Z^1(R_G)$ the following diagram is commutative:

$$\begin{tikzcd} A^1_{\mathrm{Int}\, P} \arrow[r, "r_\gamma"] \arrow[d, "\Pi_{0,1}"'] & A^1_{\mathfrak{g}} \arrow[d, "\Pi_{0,1}"] \\ A^{0,1}_{\mathrm{Int}\, P} \arrow[r, "\bar{r}_\chi"'] & A^{0,1}_{\mathfrak{g}}, \end{tikzcd} \tag{1.21}$$

where $\mathrm{Int}\, P$ is the flat group bundle corresponding to $\gamma$ and $\chi = \Pi_{0,1}\gamma$.

Now we consider the example of the abelian complex Lie group $G = \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ which will be useful in the sequel. Note that $G$ contains the compact subgroup $\mathrm{U}_1 = \{c \in \mathbb{C} \mid |c| = 1\}$.

**Example 1.1.** We want to study the homomorphism of groups $\Pi^*_{0,1} : H^1_{DR}(M, \mathbb{C}^\times) \to H^{0,1}(M, \mathbb{C}^\times)$. Consider the homomorphism $\operatorname{Exp} : \mathbb{C} \to \mathbb{C}^\times$ given by $\operatorname{Exp} c = \exp(2\pi i c)$. Clearly, $\operatorname{Exp}(\mathbb{R}) = \mathrm{U}_1$. The exact sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{\operatorname{Exp}} \mathrm{U}_1 \longrightarrow 1,$$
$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\operatorname{Exp}} \mathbb{C}^\times \longrightarrow 1$$

give rise to the following commutative diagram with exact lines:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(M,\mathbb{Z}) & \longrightarrow & H^1(M,\mathbb{R}) & \xrightarrow{\operatorname{Exp}^*} & H^1(M,\mathrm{U}_1) & \xrightarrow{\delta^*} & H^2(M,\mathbb{Z}) \\
 & & \| & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & H^1(M,\mathbb{Z}) & \longrightarrow & H^1(M,\mathbb{C}) & \xrightarrow{\operatorname{Exp}^*} & H^1(M,\mathbb{C}^\times) & \xrightarrow{\delta^*} & H^2(M,\mathbb{Z}) \\
 & & \| & & \downarrow{\scriptstyle \iota^*_{\mathbb{C}}} & & \downarrow{\scriptstyle \iota_{(\mathbb{C}^\times)^*}} & & \| \\
0 & \longrightarrow & H^1(M,\mathbb{Z}) & \longrightarrow & H^1(M,\mathcal{O}) & \xrightarrow{\operatorname{Exp}^*} & H^1(M,\mathcal{O}^\times) & \xrightarrow{\delta^*} & H^2(M,\mathbb{Z}).
\end{array}$$

Here $\delta^*$ are connecting homomorphisms of the cohomology exact sequences and arrows without denotation are given by natural inclusions of groups. It is well known that $\operatorname{Ker}\delta^* \subset H^1(M, \mathcal{O}^\times)$ is precisely the set of those holomorphic $\mathrm{GL}_1(\mathbb{C})$-bundles that are trivial as smooth principal bundles. By Propositions 1.1 and 1.4, the subgroups $\operatorname{Ker}\delta^*$ may be identified with $H^1_{DR}(M, \mathrm{U}_1)$, $H^1_{DR}(M, \mathbb{C}^\times)$ and $H^{0,1}(M, \mathbb{C}^\times)$, respectively, while $\iota_{(\mathbb{C}^\times)^*}$ identifies with $\Pi^*_{0,1}$ (see (1.19)). Using also the classical de Rham and Dolbeault isomorphisms, we obtain the following commutative diagram with exact lines:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(M,\mathbb{Z}) & \longrightarrow & H^1_{DR}(M,\mathbb{R}) & \xrightarrow{\operatorname{Exp}^*} & H^1_{DR}(M,\mathrm{U}_1) & \longrightarrow & \varepsilon \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(M,\mathbb{Z}) & \longrightarrow & H^1_{DR}(M,\mathbb{C}) & \xrightarrow{\operatorname{Exp}^*} & H^1_{DR}(M,\mathbb{C}^\times) & \longrightarrow & \varepsilon \\
 & & \| & & \downarrow{\scriptstyle \Pi^*_{0,1}} & & \downarrow{\scriptstyle \Pi^*_{0,1}} & & \\
0 & \longrightarrow & H^1(M,\mathbb{Z}) & \longrightarrow & H^1(M,\mathcal{O}) & \xrightarrow{\operatorname{Exp}^*} & H^{0,1}(M,\mathbb{C}^\times) & \longrightarrow & \varepsilon.
\end{array}$$

It follows that $H^1_{DR}(M, \mathbb{C}^\times)$ and $H^{0,1}(M, \mathbb{C}^\times)$ are connected complex abelian Lie groups, while $H^1_{DR}(M, \mathrm{U}_1)$ is a torus. The group $\operatorname{Pic} M = H^{0,1}(M, \mathbb{C}^\times)$ is, by definition, the *Picard manifold* of $M$.

The following *complement problem* seems to be important: under which conditions is the mapping $\Pi^*_{0,1} : H^1_{DR}(M, G) \to H^{0,1}(M, G)$ surjective? This is the same that for any $\alpha \in Z^1(\overline{R}_G)$, to find a form $\beta \in Z^1(R_G)$ such that $\Pi_{0,1}\beta = \alpha$. In n°2.2, we will answer this question positively in the case when $M$ is a compact Kähler manifold and $G$ is the Borel subgroup of a classical complex linear group (e.g., the complex triangular matrix group $T_n(\mathbb{C})$).

**1.4.** Let $\mathbf{V}$ be a flat complex vector bundle over a complex manifold $M$. Then we have the complexes of $\mathbf{V}$-valued forms $(A_{\mathbf{V}}, d)$, $(A^{*,q}_{\mathbf{V}}, \bar\partial)$ and $(A^{p,*}_{\mathbf{V}}, \bar\partial)$ (see n°1.2,

where we may formally set $E = \mathfrak{e} = \mathbf{V}$). Suppose that $M$ is a compact Hermitian manifold and that the structure group of $\mathbf{V}$ is $\mathrm{U}_n$. Then we can define a flat Hermitian metric on $\mathbf{V}$. This gives rise to the Laplace operators $\Delta$, $\Delta_\partial$ and $\Delta_{\bar\partial}$ in $A_{\mathbf{V}}$ given by

$$\Delta = dd^* + d^*d, \quad \Delta_\partial = \partial\partial^* + \partial^*\partial, \quad \Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial,$$

where $d^*$, $\partial^*$, $\bar\partial^*$ are formally conjugates of $d$, $\partial$, $\bar\partial$, respectively. Let us denote

$$\begin{aligned} \mathbb{H}^r &= (\operatorname{Ker}\Delta) \cap A^r_{\mathbf{V}}, \\ \mathbb{H}^{p,q}_\partial &= (\operatorname{Ker}\Delta_\partial) \cap A^{p,q}_{\mathbf{V}}, \\ \mathbb{H}^{p,q}_{\bar\partial} &= (\operatorname{Ker}\Delta_{\bar\partial}) \cap A^{p,q}_{\mathbf{V}} \end{aligned}$$

the vector spaces of harmonic forms. Any harmonic form $\omega$ is, respectively, $d$-, $\partial$- or $\bar\partial$-closed. By Hodge theorem, the correspondence $\omega \mapsto [\omega]$ gives the following isomorphisms:

$$\begin{aligned} \mathbb{H}^r &\simeq H^r_{DR}(M,\mathbf{V}) \simeq H^r(M,\mathcal{C}_{\mathbf{V}}), \\ \mathbb{H}^{p,q}_\partial &\simeq \overline{H}^{p,q}(M,\mathbf{V}) \simeq \overline{H}^q(M,\Omega^p_{\mathbf{V}}), \\ \mathbb{H}^{p,q}_{\bar\partial} &\simeq H^{p,q}(M,\mathbf{V}) \simeq H^q(M,\Omega^p_{\mathbf{V}}). \end{aligned}$$

We shall use the following fact which is well known in the case of the trivial vector bundle $\mathbf{V}$ over a compact Kähler manifold (in the more general case of a harmonic flat vector bundle a proof is outlined in [ABCKT], p. 104).

**Proposition 1.7.** *Suppose that $M$ is a compact Kähler manifold. Then*

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar\partial}.$$

*It follows that*

$$\mathbb{H}^r = \bigoplus_{p+q=r} \mathbb{H}^{p,q}_{\bar\partial}, \quad \mathbb{H}^{p,q}_{\bar\partial} = \mathbb{H}^{p,q}_{\partial}.$$

In what follows, we will omit the superscripts $\partial$ and $\bar\partial$ in the notation of harmonic forms.

**Corollary 1.** *Under the above conditions, the vector spaces $\mathbb{H}^{p,0}$ and $\mathbb{H}^{0,q}$ coincide with those of holomorphic $\mathbf{V}$-valued $p$-forms and antiholomorphic $\mathbf{V}$-valued $q$-forms, respectively.*

**Corollary 2.** *Under the above conditions,*

$$\begin{aligned} H^r_{DR}(M,\mathbf{V}) &\simeq \bigoplus_{p+q=r} H^{p,q}(M,\mathbf{V}) \simeq \bigoplus_{p+q=r} \overline{H}^{p,q}(M,\mathbf{V}), \\ H^r(M,\mathcal{C}_{\mathbf{V}}) &\simeq \bigoplus_{p+q=r} H^p(M,\Omega^q_{\mathbf{V}}) \simeq \bigoplus_{p+q=r} H^p(M,\overline{\Omega}^q_{\mathbf{V}}). \end{aligned}$$

Arguing as in the proof of the lemma in [GH], Ch.1, Section 2 (see also [DGMS]), and applying Proposition 1.7, we obtain the following '$\partial\bar\partial$-lemma':

**Corollary 3.** *Let $\mathbf{V}$ be a flat vector bundle with structure group $\mathrm{U}_n$ over a compact Kähler manifold. Suppose that $\omega \in A^{p,q}_{\mathbf{V}}$ is d-closed and $\partial$- or $\bar\partial$-exact, where $p, q \geq 1$. Then there exists $\psi \in A^{p-1,q-1}_{\mathbf{V}}$ such that $\omega = \partial\bar\partial\psi$.*

Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a linear complex Lie group and $z = (z_{ij}) \in Z^1(\mathfrak{U}, G)$. Consider the flat group bundle $E$ corresponding to the cocycle $\operatorname{Int} z$ (see n°1.1). Clearly, $z$ determines a flat vector bundle $\mathbf{V}$ with fibre $\mathbb{C}^n$. Then $E$ can be regarded as a subbundle of the flat vector bundle $\mathbf{End}\mathbf{V}$ with fibre $\operatorname{End}\mathbb{C}^n$ which is obtained by twisting $M \times \operatorname{End}\mathbb{C}^n$ with the help of $\operatorname{Int} z$. The group $F_E$ will coincide with a subgroup of the group $\operatorname{Aut}\mathbf{V}$ of automorphisms of $\mathbf{V}$.

**Corollary 4.** *Suppose that $z_{ij} \in \mathrm{U}_n$ and that $M$ is a compact Kähler manifold. If $a \in F_E$ satisfies $\Pi_{1,0}\delta_0(a) = 0$ or $\Pi_{0,1}\delta_0(a) = 0$, then $\delta_0(a) = 0$, i.e., $a$ is a flat section.*

*Proof.* Apply Corollary 1 to the vector bundle $\mathbf{End}\mathbf{V}$.

We return now to the example $G = \mathbb{C}^\times$ (see Section 1.2). Using Hodge theory (in the case of the trivial bundle $M \times \mathbb{C}$), we prove the following

**Proposition 1.8.** *Let $M$ be a compact Kähler manifold. The mapping $\Pi^*_{0,1} : H^1_{DR}(M, \mathrm{U}_1) \to H^{0,1}(M, \mathbb{C}^\times)$ is an isomorphism of real Lie groups.*

*Proof.* Due to the latter diagram of Example 1.1, it suffices to prove that $\Pi^*_{0,1} : H^1_{DR}(M, \mathbb{R}) \to H^{0,1}(M, \mathbb{C})$ is an isomorphism of real vector spaces. But this is a well-known consequence of Proposition 1.7. The inverse to $\Pi^*_{0,1}$ is given by $\omega \mapsto \omega + \bar\omega$, $\omega \in \mathbb{H}^{0,1}$.

As a corollary, we get the following well-known fact: for any compact Kähler manifold $M$, the group $\operatorname{Pic} M$ is a complex torus.

The above proof shows that any class from $H^1_{DR}(M, \mathrm{U}_1)$, $H^1_{DR}(M, \mathbb{C}^\times)$ or $H^{0,1}(M, \mathbb{C}^\times)$ contains a harmonic form. On the cochain level, the inverse to $\Pi^*_{0,1} : H^1_{DR}(M, \mathrm{U}_1) \to H^{0,1}(M, \mathbb{C}^\times)$ can be given by $\omega \mapsto \omega - \bar\omega$, $\omega \in \mathbb{H}^{0,1}$.

**1.5.** Suppose that we have the semi-direct product $G = B \rtimes A$ of complex Lie groups. Here $B$ may be regarded as a normal Lie subgroup and $A$ as a Lie subgroup of $G$. Denote by $p : G \to A$ the natural projection homomorphism and by $q : A \to G$ the natural inclusion. Then $B = \operatorname{Ker} p$ and $p \circ q = \mathrm{id}$. Clearly, we have the semi-direct decomposition $\mathfrak{g} = \mathfrak{b} \niplus \mathfrak{a}$, where $\mathfrak{b}$ is an ideal and $\mathfrak{a}$ a subalgebra of $\mathfrak{g}$. It follows that $A_{\mathfrak{g}} = A_{\mathfrak{b}} \niplus A_{\mathfrak{a}}$ for an arbitrary manifold $M$. Any form $\omega \in A^p_{\mathfrak{g}}$ on $M$ is uniquely decomposed as $\omega = \omega_{\mathfrak{b}} + \omega_{\mathfrak{a}}$, where $\omega_{\mathfrak{b}} \in A^p_{\mathfrak{b}}$ and $\omega_{\mathfrak{a}} = dp \circ \omega \in A^p_{\mathfrak{a}}$.

The relations between the de Rham (and Dolbeault) cohomology with values in $G$, $A$, $B$ are described in [O4], Theorems 2.2 and 2.4. In particular, there is the following commutative diagram with exact lines:

$$
\begin{tikzcd}
H^1_{DR}(M,B) \arrow[r, "i_1^*"] \arrow[d, "\Pi^*_{0,1}"'] & H^1_{DR}(M,G) \arrow[r, "p_1^*"] \arrow[d, "\Pi^*_{0,1}"'] & H^1_{DR}(M,A) \arrow[r] \arrow[d, "\Pi^*_{0,1}"] & \varepsilon \\
H^{0,1}(M,B) \arrow[r, "i_1^*"'] & H^{0,1}(M,G) \arrow[r, "p_1^*"'] & H^{0,1}(M,A) \arrow[r] & \varepsilon.
\end{tikzcd}
\tag{1.22}
$$

Both mappings $i_1^*$ are induced by the natural inclusion $i : B \to G$. We also have the injective mappings $q_1^* : H^1_{DR}(M,A) \to H^1_{DR}(M,G)$ and $q_1^* : H^{0,1}(M,A) \to$

$H^{0,1}(M,G)$ such that $p_1^* \circ q_1^* = \mathrm{id}$. We will identify $H^1_{DR}(M,A)$ and $H^{0,1}(M,A)$ with their images in $H^1_{DR}(M,G)$ and $H^{0,1}(M,G)$, respectively.

Suppose now that we are given a form $\gamma \in Z^1(R_A)$. Denote by $\xi \in H^1_{DR}(M,A)$ and $\zeta \in H^{0,1}(M,A)$ the cohomology classes of $\gamma$ and $\chi = \Pi_{0,1}\gamma$. Then $\Pi^*_{0,1}\xi = \zeta$. We want to describe the subsets $(p_1^*)^{-1}(\xi) \subset H^1_{DR}(M,G)$ and $(p_1^*)^{-1}(\zeta) \subset H^{0,1}(M,G)$ using splitting constructions. As above, choose an open cover $\mathfrak{U} = (U_i)$ of $M$ such that $\gamma = \delta_0(a_i)$ in $U_i$, where $a_i \in \Gamma(U_i, \mathcal{F}_A)$, and consider the 1-cocycle $z = (z_{ij}) \in Z^1(\mathfrak{U}, A)$ given by $z_{ij} = a_i^{-1}a_j$. Denote by $E$ the flat group bundle with the $G$ determined by the cocycle $(\mathrm{Int}\, z_{ij})$. Since $z_{ij} \in A$, we have the semi-direct decomposition $E = E_B \rtimes E_A$, where $E_B$ and $E_A$ are group subbundles with fibres $B$ and $A$, respectively. We also consider the corresponding Lie algebra bundle $\mathfrak{e} = \mathfrak{e}_{\mathfrak{b}} \niplus \mathfrak{e}_{\mathfrak{a}}$. Then we have the following commutative diagram with exact lines:

$$
\begin{tikzcd}
H^1_{DR}(M,E_B) \arrow[r, "i_1^*"] \arrow[d, "\Pi^*_{0,1}"'] & H^1_{DR}(M,E) \arrow[r, "p_1^*"] \arrow[d, "\Pi^*_{0,1}"'] & H^1_{DR}(M,E_A) \arrow[r] \arrow[d, "\Pi^*_{0,1}"] & \varepsilon \\
H^{0,1}(M,E_B) \arrow[r, "i_1^*"'] & H^{0,1}(M,E) \arrow[r, "p_1^*"'] & H^{0,1}(M,E_A) \arrow[r] & \varepsilon.
\end{tikzcd}
\tag{1.23}
$$

Its right square is got from the right square of (1.22) by applying to the lines the bijections $r_\gamma^{-1}$ and $\bar{r}_\chi^{-1}$ (see (1.21)). Thus, $(p_1^*)^{-1}(\xi)$ is identified with $i_1^*(H^1_{DR}(M,E_B))$, while $(p_1^*)^{-1}(\zeta)$ is identified with $i_1^*(H^{0,1}(M,E_B))$.

## 2. Cohomology with values in certain solvable algebraic groups

**2.1.** Let $G$ be a connected solvable complex linear algebraic group. It is well known (see, e.g., [Hu], Ch. VII) that $G$ admits the semi-direct decomposition $G = N \rtimes S$, where $N$ is a normal unipotent algebraic subgroup (the unipotent radical) of $G$ and $S \simeq (\mathbb{C}^\times)^n$ an algebraic torus. Respectively, for the tangent Lie algebra $\mathfrak{g}$ of $G$ we have the semi-direct decomposition $\mathfrak{g} = \mathfrak{n} \niplus \mathfrak{s}$, where $\mathfrak{n}$ and $\mathfrak{s}$ are the ideal and the subalgebra of $\mathfrak{g}$ corresponding to $N$ and $S$, respectively. Denote by $p : G \to S$ the natural projection and by $q : S \to G$ the natural inclusion.

Let $K$ be the compact real form of $S$. Clearly, $K \simeq \mathrm{U}_1^n$. We will also consider the real algebraic subgroup $G_K = N \rtimes K$ of $G$.

The most important example is the subgroup $G = T_n(\mathbb{C})$ of $\mathrm{GL}_n(\mathbb{C})$ consisting of upper triangular matrices. In this case $N = T_n^0(\mathbb{C})$ is the normal subgroup of unipotent upper triangular matrices and $S = D_n(\mathbb{C})$ the subgroup of diagonal matrices which is naturally identified with $(\mathbb{C}^\times)^n$. The Lie algebra $\mathfrak{g} = \mathfrak{t}_n(\mathbb{C})$ is the subalgebra of upper triangular matrices in $\mathfrak{gl}_n(\mathbb{C})$. Also $\mathfrak{n}$ is the ideal of nilpotent upper triangular matrices and $\mathfrak{s}$ is the subalgebra of diagonal matrices which is naturally identified with $\mathbb{C}^n$. Further, $K$ is the subgroup of unitary diagonal matrices, identified with $\mathrm{U}_1^n$, and its Lie subalgebra $\mathfrak{k} \subset \mathfrak{s}$ is the subalgebra of pure imaginary diagonal matrices. The de Rham complex $A_{\mathfrak{u}}$ is identified with the complex of $\mathfrak{s}$-valued forms $\gamma$ satisfying $\overline{\gamma} = -\gamma$. The subgroup $G_K = T_n(\mathbb{C})_K$ of $T_n(\mathbb{C})$ is formed by matrices with unitary diagonal.

Let $G$ be an arbitrary connected solvable complex linear algebraic group and suppose that $V$ is an algebraic $G$-module, i.e., that a polynomial representation $\phi : G \to \mathrm{GL}(V)$ is given. Let $\mathfrak{U}$ be an open cover of a complex manifold $M$ and $\gamma \in Z^1(\mathfrak{U}, G_{\omega})$. Consider the corresponding flat vector bundle $\mathbf{V}(\gamma)$ of rank $n$ over

$M$ with structure group $G_K$. Then there is the de Rham complex $(A_{\mathbf{V}(z)}, d)$, as well as the Dolbeault complexes $(A^{p,*}_{\mathbf{V}(z)}, \bar\partial)$ and $(A^{*,q}_{\mathbf{V}(z)}, \partial)$.

The projection $p : G_K \to K$ gives rise to the mapping $p_1^* : H^1(M, G_K) \to H^1(M, K)$. It can be easily described on the cocycle level. Namely, for the above cocycle $z = (z_{ij})$ we have $z_{ij} = u_{ij} n_{ij}$, where $u_{ij} \in K$, $n_{ij} \in N$. Since $u_{ij} = p(z_{ij})$, we see that $u \in Z^1(\mathfrak{U}, K)$ and $p_1^*$ sends the cohomology class of $z$ to that of $u$.

Suppose now that $z_{ij} = g_i^{-1} g_j$ in $U_i \cap U_j \neq \emptyset$, where $g_i : U_i \to G_K$ are smooth functions. Then $\mathbf{V}(z)$ is trivial as a smooth vector bundle with structure group $G_K$. We have $g_i = u_i n_i$ with $u_i : U_i \to K$, $n_i : U_i \to N$, whence

$$\begin{aligned} u_{ij} &= p(g_i)^{-1} p(g_j) = u_i^{-1} u_j, \\ z_{ij} &= n_i^{-1} u_{ij} n_j, \end{aligned} \tag{2.1}$$

In particular, we see that the 0-cochain $(n_i)$ determines an isomorphism of the flat vector bundles $\mathbf{V}(z)$ and $\mathbf{V}(u)$ in the category of smooth vector bundles with structure group $G_K$. In the next lemma, this cochain should satisfy the condition $\Pi_{0,1}\delta_0(n_i) = 0$ for all $i$ which means that the corresponding isomorphism is antiholomorphic.

**Lemma 2.1.** *Let $z \in Z^1(\mathfrak{U}, G_K)$ be a Čech cocycle on a complex manifold $M$ satisfying the triviality condition described above. Suppose that $M$ is a compact Kähler manifold and that $\Pi_{0,1}\delta_0(n_i) = 0$ for any $i$. Then*

(i) *Any class from $H^{0,q}(M, \mathbf{V}(z))$, $q \geq 0$, contains a $d$-closed $\mathbf{V}(z)$-valued $(0, q)$-form;*

(ii) *Any holomorphic $\mathbf{V}(z)$-valued form $\alpha$ is $d$-closed. If, in addition, $\alpha$ is $\partial$-exact, then $\alpha = 0$;*

(iii) *If $\alpha \in A^{p,0}_{\mathbf{V}(z)}$ satisfies $\partial\alpha = 0$ and $\bar\partial\alpha = \partial\beta$ for a certain $\beta \in A^{p-1,1}_{\mathbf{V}(z)}$, then there exists $\gamma \in A^{p-1,0}_{\mathbf{V}(z)}$ such that $\alpha + \partial\gamma$ is holomorphic.*

*Proof.* We will use the induction on $n = \dim V$. Assume that $n = 1$. Clearly, $\phi(G_K) \subset U_1$, and hence $\mathbf{V}(z)$ is the flat line bundle determined by the cocycle $(\phi(z_{ij}))$, where $\phi(z_{ij}) \in \mathrm{U}_1$. Therefore we can apply Hodge theory discussed in Section 1.4. Clearly, (i) is implied by the fact that any Dolbeault cohomology class contains a harmonic form (see Proposition 1.7), and (ii) follows from Corollary 1 of Proposition 1.7. To prove (iii), we note that $d\partial\beta = d\bar\partial\alpha = 0$. Applying Corollary 3 of the same proposition, we see that $\partial\beta = \bar\partial\partial\gamma$, whence $\bar\partial(\alpha - \partial\gamma) = 0$.

To argue by induction, we need the following construction. By Lie theorem, there exists a basis $e_1, \dots, e_n$ of $V$ such that $\phi$ is expressed by the matrix

$$\phi(g) = \begin{pmatrix} \phi_0(g) & * \\ 0 & \tilde\phi(g) \end{pmatrix}, \quad g \in G,$$

where $\tilde\phi : G \to T_{n-1}(\mathbb{C})$ is a representation satisfying $\tilde\phi(g) \in D_{n-1}(\mathbb{C})$ for $g \in S$, and hence $\tilde\phi(g) \in \mathrm{U}_1^{n-1}$, for $g \in K$. The exact sequence of $G$-modules

$$0 \longrightarrow V^0 \longrightarrow V \longrightarrow \tilde V \longrightarrow 0,$$

where $V^0 = \langle e_1 \rangle$, $\tilde V = \langle e_2, \dots, e_n \rangle$, gives rise to the following two exact sequences of flat vector bundles:

$$0 \longrightarrow \mathbf{V}^0(z) \xrightarrow{\lambda} \mathbf{V}(z) \xrightarrow{\varkappa} \tilde{\mathbf{V}}(z) \longrightarrow 0, \tag{2.2}$$

$$0 \longrightarrow \mathbf{V}^0(u) \xrightarrow{\lambda} \mathbf{V}(u) \xrightarrow{\varkappa} \tilde{\mathbf{V}}(u) \longrightarrow 0 \tag{2.3}$$

Since the matrices $\phi(u_{ij})$ are diagonal, the sequence (2.3) is split.

From (2.2) we obtain, for any $q \geq 0$, the following exact sequence of sheaves of antiholomorphic $q$-forms:

$$0 \longrightarrow \overline{\Omega}^q_{\mathbf{V}^0(z)} \xrightarrow{\lambda} \overline{\Omega}^q_{\mathbf{V}(z)} \xrightarrow{\varkappa} \overline{\Omega}^q_{\tilde{\mathbf{V}}(z)} \longrightarrow 0. \tag{2.4}$$

The mappings $n_i$ (see (2.1)) determine an antiholomorphic isomorphism of (2.2) onto the split exact sequence (2.3). It follows that the exact sequence of sheaves (2.4) is split. Hence, for any $p \geq 0$, we get the following split exact sequence of cohomology groups:

$$0 \longrightarrow H^p(M, \overline{\Omega}^q_{\mathbf{V}^0(z)}) \xrightarrow{\lambda^*} H^p(M, \overline{\Omega}^q_{\mathbf{V}(z)}) \xrightarrow{\varkappa^*} H^p(M, \overline{\Omega}^q_{\tilde{\mathbf{V}}(z)}) \longrightarrow 0. \tag{2.5}$$

Using the isomorphism (1.18), we can interprete (2.5) as the following split exact sequence:

$$0 \longrightarrow \overline{H}^{p,q}(M, \mathbf{V}^0(z)) \xrightarrow{\lambda^*} \overline{H}^{p,q}(M, \mathbf{V}(z)) \xrightarrow{\varkappa^*} \overline{H}^{p,q}(M, \tilde{\mathbf{V}}(z)) \longrightarrow 0. \tag{2.6}$$

Let us also note that (2.2) gives rise to the following split exact sequences:

$$0 \longrightarrow A^{p,q}_{\mathbf{V}^0(z)} \xrightarrow{\lambda} A^{p,q}_{\mathbf{V}(z)} \xrightarrow{\varkappa} A^{p,q}_{\tilde{\mathbf{V}}(z)} \longrightarrow 0. \tag{2.7}$$

Now we pass to the induction argument. Suppose that (i), (ii) and (iii) are true for vector bundles of rank $n-1$.

(i) We will assume that $q \geq 1$, since (i) coincides with (ii) in the case $q = 0$. Let $\alpha$ be an $\mathbf{V}(z)$-valued $(0,q)$-form such that $\bar{\partial}\alpha = 0$, where $\dim V = n$. Consider the exact sequence (2.7) for $p = 0$. By induction hypothesis, $\varkappa(\alpha) = \beta + \bar{\partial}\varphi$, where $d\beta = 0$ and $\varphi \in A^{0,q-1}_{\tilde{\mathbf{V}}(z)}$. Then $\varphi = \varkappa(\psi)$ for a certain $\psi \in A^{0,q-1}_{\mathbf{V}(z)}$. Therefore $\beta = \varkappa(\alpha - \bar{\partial}\psi)$. Hence $\varkappa(\partial(\alpha - \bar{\partial}\psi)) = 0$, and $\partial(\alpha - \bar{\partial}\psi) = \lambda(\alpha')$, where $\alpha'$ is a $d$-closed $\mathbf{V}^0(z)$-valued $(1,q)$-form. Using (2.6), we conclude that $\alpha'$ is

$\partial$-exact. Applying Corollary 3 of Proposition 1.7, we see that $\alpha' = \partial\bar{\partial}\omega$ for a certain $\omega \in A^{0,q-1}_{\mathbf{V}^0(z)}$. It follows that $\partial(\alpha - \bar{\partial}(\psi + \lambda(\omega))) = 0$. Thus, $\alpha - \bar{\partial}(\psi + \lambda(\omega))$ is the desired $d$-closed form.

(ii) Let $\alpha$ be a holomorphic $\mathbf{V}(z)$-valued $p$-form, where $\dim V = n$. By induction hypothesis, $d\varkappa(\alpha) = 0$, whence $\varkappa(d\alpha) = \varkappa(\partial\alpha) = 0$. It follows that $\partial\alpha = \lambda(\beta)$, where $\beta$ is a $\mathbf{V}^0(z)$-valued holomorphic $(p+1)$-form. From (2.6) we conclude that $\beta$ is $\partial$-exact. Using Corollary 1 of Proposition 1.7 applied to $\mathbf{V}^0(z)$-valued forms, we see that $\beta$ is harmonic, and hence $\beta = 0$. Therefore $\partial\alpha = d\alpha = 0$.

Suppose that $\alpha = \partial\gamma$. Then $\varkappa(\alpha) = \partial\varkappa(\gamma) = 0$ by induction hypothesis. Therefore $\alpha = \lambda(\beta)$, where $\beta$ is a $\mathbf{V}^0(z)$-valued holomorphic $p$-form. Then (2.6) implies that $\beta$ is $\partial$-exact in $A_{\mathbf{V}^0(z)}$, and hence $\beta = 0$.

(iii) Suppose that $\partial\alpha = 0$ and $\bar{\partial}\alpha = \partial\beta$. Clearly, $\varkappa(\alpha)$ satisfies similar conditions. By induction hypothesis, $\bar{\partial}(\varkappa(\alpha) + \partial\gamma_1) = 0$ for a certain $\gamma_1 \in A^{p-1,0}_{\tilde{\mathbf{V}}(z)}$. By (2.7), we may choose $\gamma_2 \in A^{p-1,0}_{\mathbf{V}(z)}$ such that $\varkappa(\gamma_2) = \gamma_1$. Then $\varkappa(\bar{\partial}(\alpha + \partial\gamma_2)) = 0$, whence $\bar{\partial}(\alpha + \partial\gamma_2) = \lambda(\varphi)$ for a $\mathbf{V}^0(z)$-valued $(p,1)$-form $\varphi$. Clearly, $d\varphi = 0$, and (2.6) implies that $\varphi$ is $\partial$-exact. By Corollary 3 of Proposition 1.7, $\varphi = \bar{\partial}\partial\gamma_3$. Then $\bar{\partial}(\alpha + \partial(\gamma_2 - \gamma_3)) = 0$.

**Corollary.** *Let $E$ be the flat group bundle with fibre $G$ determined by the cocycle* $\operatorname{Int} z$. *If $a \in F_E$ is holomorphic, then $a$ is flat.*

*Proof.* The argument is similar to the proof of Corollary 4 of Proposition 1.7. Namely, we fix a faithful holomorphic representation $\phi$ of $G$ and consider the corresponding flat vector bundle $\mathbf{V}(z)$. Then $E$ is a subbundle of the flat vector bundle $\mathbf{End}\,\mathbf{V}(z)$ determined by the cocycle $\operatorname{Int}\phi(z)$. Then we apply the statement (ii) to the latter flat vector bundle.

**2.2.** Let $M$ be a compact Kähler manifold. Then any class from $H^{0,1}(M,\mathbb{C})$ contains a $d$-closed (i.e., antiholomorphic) $(0,1)$-form. In this subsection, we consider a non-abelian analogue of this fact involving a class of solvable algebraic groups. This study is closely related to the complement problem mentioned in n°1.3.

We retain the notation of n°2.1. Let $G = N \rtimes S$ be a connected solvable complex linear algebraic group. Then we have the following commutative diagram (see (1.22)):

$$
\begin{tikzcd}
H^1_{DR}(M,G) \arrow[r, "p_1^*"] \arrow[d, "\Pi^*_{0,1}"'] & H^1_{DR}(M,S) \arrow[d, "\Pi^*_{0,1}"] \\
H^{0,1}(M,G) \arrow[r, "p_1^*"'] & H^{0,1}(M,S).
\end{tikzcd}
\tag{2.8}
$$

As we saw in n°1.5, both $p_1^*$ are surjective, and $H^1_{DR}(M,S)$ and $H^{0,1}(M,S)$ are naturally embedded in $H^1_{DR}(M,G)$ and $H^{0,1}(M,G)$, respectively. Since $K$ is a direct factor of $S$, we also have an embedding $H^1_{DR}(M,K) \subset H^1_{DR}(M,S)$.

Due to Proposition 1.8, $\Pi^*_{0,1}$ induces an isomorphism of groups $H^1_{DR}(M,U) \to H^{0,1}(M,S)$ (which gives the solution of the complement problem in the case $G = S$). It can be expressed by the isomorphism of the spaces of $\mathfrak{k}$-valued and $\mathfrak{s}$-valued harmonic forms $\mathbb{H}^1_{\mathfrak{k}} \to \mathbb{H}^{0,1}_{\mathfrak{s}}$ given by $-\bar\chi + \chi \mapsto \chi$. Let us fix a class $\zeta \in H^{0,1}(M,S)$ and a harmonic form $\chi \in A^{0,1}_{\mathfrak{s}}$ representing this class, and denote $\gamma = -\bar\chi + \chi$. Choose an open cover $\mathfrak{U} = (U_i)$ of $M$ such that $\gamma = \delta_0(u_i)$ in any $U_i$ for certain smooth functions $u_i : U_i \to K$, and construct the Čech cocycle $z = (z_{ij})$, where $z_{ij} = u_i^{-1}u_j$ in any $U_i \cap U_j \neq \emptyset$. We fix the functions $u_i$, and hence the cocycle $z$, as well.

We will use the twisted complex $R_E$ corresponding to $\zeta$. Here we denote by $E$ the flat group bundle obtained from $M \times G$ by twisting with the help of $\operatorname{Int} z$, and by $\mathfrak{e}$ the corresponding Lie algebra bundle (see n°1.1). Since $z$ takes its values in $K$, we also get the normal group subbundle $E_N \subset E$ with fibre $N$ corresponding to the cocycle $\operatorname{Int} z|N$ such that $E$ is the semi-direct product of $N$ and of the trivial bundle $M \times S$. Respectively, we get the semi-direct decomposition $A_{\mathfrak{e}} = A_{\mathfrak{e}_{\mathfrak{n}}} \Supset A_{\mathfrak{s}}$.

We also need the mapping $r_\gamma : A^1_{\mathfrak{e}} \to A^1_{\mathfrak{g}}$ introduced in n°1.1. By Proposition 1.3, it induces the bijection $r^*_\gamma : H^1_{DR}(M,E) \to H^1_{DR}(M,G)$ such that $r^*_\gamma(\varepsilon) = \xi$ is the cohomology class of $\gamma$. Similarly (see Proposition 1.6), we have the mapping $\bar r_\chi : A^{0,1}_{\mathfrak{e}} \to A^{0,1}_{\mathfrak{g}}$ inducing the bijection $\bar r^*_\chi : H^{0,1}(M,E) \to H^{0,1}(M,G)$ such that $\bar r^*_\chi(\varepsilon) = \zeta$.

Note that we have the following commutative diagram with exact lines (see

(1.23)):

$$
\begin{array}{ccccccc}
H^1_{DR}(M,E_N) & \xrightarrow{i_1^*} & H^1_{DR}(M,E) & \xrightarrow{p_1^*} & H^1_{DR}(M,S) & \longrightarrow & \varepsilon \\
\Pi^*_{0,1}\big\downarrow & & \Pi^*_{0,1}\big\downarrow & & \big\downarrow\Pi^*_{0,1} & & \\
H^{0,1}(M,E_N) & \xrightarrow[i_1^*]{} & H^{0,1}(M,E) & \xrightarrow[p_1^*]{} & H^{0,1}(M,S) & \longrightarrow & \varepsilon.
\end{array}
\tag{2.9}
$$

Here both $p_1^*$ are surjective, and $H^1_{DR}(M,S)$ and $H^{0,1}(M,S)$ are naturally embedded in $H^1_{DR}(M,E)$ and $H^{0,1}(M,E)$, respectively. As we saw in n°1.5, its right square is got from (1.22) by applying to the lines the bijections $r_\gamma^{-1}$ and $\bar{r}_\chi^{-1}$.

We will say that $G$ *has the Hodge property* if for any compact Kähler manifold $M$ and for any group bundle $E$ over $M$ of the class described above, the following is satisfied: for each class $\sigma \in H^{0,1}(M,E)$ such that $p_1^*(\sigma) = \varepsilon$ there exists a form $\eta \in \sigma$ with the properties

$$\eta \in A^{0,1}_{\mathfrak{e}_\mathfrak{n}}, \quad \delta_1(\eta) = 0. \tag{2.10}$$

Suppose that $\eta$ satisfies (2.10) and denote $\beta = r_\gamma(\eta) \in Z^1(R_G)$. Then $\beta = \operatorname{Ad} u_i(\eta_i) + \gamma$, whence

$$\beta_\mathfrak{s} = \gamma, \quad \Pi_{0,1}\beta = \beta_\mathfrak{n} + \chi. \tag{2.11}$$

Taking into account the remarks made above, we can reformulate our definition in terms of the complexes $R_G$ and $\overline{R}_G$. Namely, $G$ has the Hodge property if for any compact Kähler manifold $M$ and for any $\mathfrak{k}$-valued harmonic 1-form $\gamma$, the following is valid: for each class $\tilde{\sigma} \in H^{0,1}(M,G)$ such that $p_1^*(\tilde{\sigma}) = [\Pi_{0,1}\gamma]$, there exists a form $\beta \in Z^1(R_G)$ satisfying (2.11).

In particular, we see that the Hodge property implies the positive solution of the complement problem for the group $G$ over a compact Kähler manifold.

**Example 2.1.** Suppose that the unipotent radical $N$ of $G$ is abelian. Then $E$ is a flat vector bundle with unitary structure group. Hence, Proposition 1.7 implies that $G$ has the Hodge property.

Take a form $\eta \in R^1_E$ satisfying (2.10). Then the form $\beta = r_\gamma(\eta)$ can be used, instead of $\gamma$, for twisting the complex $R_G$. To do this, we write $\beta = \delta_0(b_i)$ for a certain smooth function $b_i : U_i \to G$ in any $U_i$. These functions can be chosen in such a way that $b_i = u_i n_i$ for certain smooth functions $n_i : U_i \to N$. In fact, $\eta_i = \rho(u_i)^{-1}(\beta_i) = \rho(u_i)^{-1}(\delta_0(b_i)) = \delta_0(u_i^{-1}b_i)$. On the other hand, $\eta_i = \delta_0(n_i)$ for certain smooth functions $n_i : U_i \to N$. Therefore $n_i = u_i^{-1}b_i g_i$, where $g_i : U_i \to G$ satisfy $\delta_0(g) = 0$. Then $b_i g_i = u_i n_i$ and $\delta_0(b_i g_i) = 0$. Replacing $b_i$ by $b_i g_i$, we get the desired result. Then we construct the Čech cocycle $w = (w_{ij}) \in Z^1(\mathfrak{U}, G)$, where

$$w_{ij} = b_i^{-1} b_j = n_i^{-1} z_{ij} n_i.$$

Consider the flat group bundle $E'$ and the flat Lie algebra bundle $\mathfrak{e}'$ with fibres $G$ and $\mathfrak{g}$ determined by the cocycles $(\operatorname{Int} w_{ij})$ and $(\operatorname{Ad} w_{ij})$, respectively. Clearly, $\beta = \delta_0(u_i n_i) = \gamma + \operatorname{Ad} u_i(\delta_0(n_i))$, whence

$$\eta_i = \operatorname{Ad} u_i^{-1}(\beta_i - \gamma) = \delta_0(n_i).$$

and $\Pi_{1,0}\delta_0(n_i) = 0$. Therefore $\mathfrak{e}'$ satisfies the conditions of Lemma 2.1.

We will also use the corresponding mapping $r_\beta : A^1_{\mathfrak{e}'} \to A^1_{\mathfrak{g}}$. Let us find an explicit expression for the mapping $r_\beta^{-1} \circ r_\gamma : \psi \mapsto \psi'$ of the space $A^1_{\mathfrak{e}}$ onto $A^1_{\mathfrak{e}'}$. Clearly, $\rho(u_i)(\psi_i) = r_\gamma(\psi) = r_\beta(\psi') = \rho(u_i n_i)(\psi'_i)$, whence

$$\begin{aligned} \psi_i &= \rho(n_i)\psi'_i, \\ \psi'_i &= \operatorname{Ad} n_i^{-1}(\psi_i - \eta_i). \end{aligned} \tag{2.12}$$

Now we are going to prove the Hodge property for certain solvable algebraic groups. The main role will play the following result.

**Lemma 2.2.** *Suppose that $G = B \rtimes A$, where $B$ is an abelian unipotent normal algebraic subgroup and $A$ an algebraic subgroup having the Hodge property. Then $G$ has the Hodge property.*

*Proof.* We have the semi-direct decomposition $A = N_0 \rtimes S$, where $N_0$ is the unipotent radical of $A$ and $S$ is an algebraic torus. Then $G = N \rtimes S$, where $N = B \rtimes N_0$. Clearly, $N$ is a unipotent algebraic subgroup, and hence the unipotent radical of $G$.

Now let $M$ be a compact Kähler manifold. Fix a class $\zeta \in H^{0,1}(M, S)$ and a harmonic form $\chi \in A^{0,1}_{\mathfrak{s}}$ representing this class, denote $\gamma = -\bar\chi + \chi$ and construct the cocycle $z$ and the flat group bundle $E$ over $M$ as described above. Choose a class $\sigma \in H^{0,1}(M, E)$ such that $p_1^*(\sigma) = \varepsilon$. Then $\sigma = [\alpha]$ for a certain $\alpha \in Z^1(\overline{R}_{E_N})$. Since $A$ and $B$ are invariant under $\operatorname{Int} z_{ij}$, we have $E_N = E_B \rtimes E_{N_0}$, where $E_B$ and $E_{N_0}$ are the subbundles corresponding to the subgroups $B$ and $N_0$ of $N$. Respectively, $A^{0,1}_{\mathfrak{e}_\mathfrak{n}} = A^{0,1}_{\mathfrak{e}_\mathfrak{b}} \oplus A^{0,1}_{\mathfrak{e}_{\mathfrak{n}_0}}$, where $\mathfrak{e}_\mathfrak{b}$ and $\mathfrak{e}_{\mathfrak{n}_0}$ are the corresponding subbundles of the Lie algebra bundle $\mathfrak{e}_\mathfrak{n}$. Hence $\alpha = \alpha_\mathfrak{b} + \alpha_{\mathfrak{e}_{\mathfrak{n}_0}}$, where $\alpha_\mathfrak{b} \in A^{0,1}_{\mathfrak{e}_\mathfrak{b}}$, while $\alpha_{\mathfrak{e}_{\mathfrak{n}_0}} \in Z^1(\overline{R}_{E_{N_0}})$. Since $A$ has the Hodge property, $\alpha_{\mathfrak{e}_{\mathfrak{n}_0}}$ is cohomologous in $\overline{R}_{E_A}$ to a form $\eta_\mathfrak{a} \in Z^1(R_{E_{N_0}})$. Replacing $\alpha$ by a cohomologous cocycle, we may assume that $\alpha_{\mathfrak{e}_{\mathfrak{n}_0}} = \eta_\mathfrak{a}$.

Consider now the form $\beta_\mathfrak{a} = r_\gamma(\eta_\mathfrak{a}) \in Z^1(R_A)$. Clearly, $\eta_\mathfrak{a} \in R^1_E$ satisfies (2.10). Hence we may apply the construction of the flat group bundle $E'$ with fibre $G$ given above using the form $\beta_\mathfrak{a}$ as $\beta$. In particular, we can write $\beta_\mathfrak{a} = \delta_0(a_i)$ in any $U_i$, where $a_i = u_i n_i$ for certain smooth functions $n_i : U_i \to N_0$ satisfying $\Pi_{1,0}\delta_0(n_i) = 0$. Then $E'$ is determined by the cocycle $w \in Z^1(\mathfrak{U}, N_0 \rtimes K)$ given by $w_{ij} = a_i^{-1} a_j$. Since $w_{ij} \in A$, we have $E' = E'_B \rtimes E'_A$, where $E'_B$ and $E'_A$ are the subbundles corresponding to the subgroups $B$ and $A$ of $G$. Using (2.12), we get

$$\alpha' = \operatorname{Ad} n_i^{-1}(\alpha_i - \eta_i) = \operatorname{Ad} n_i^{-1}((\alpha_\mathfrak{b})_i),$$

whence

$$\bar r^{-1}_{\Pi_{0,1}\beta_\mathfrak{a}}(\bar r_\chi(\alpha)) = \Pi_{0,1}\alpha' = \alpha' = \operatorname{Ad} n_i^{-1}(\alpha_\mathfrak{b})_i).$$

Thus, $\alpha' \in Z^1(\overline{R}_{E'_B})$. Since $B$ is abelian, $E'_B$ is a flat vector bundle determined by the cocycle $\operatorname{Int} w_{ij}|B$ and hence satisfying the conditions of Lemma 2.1. By the assertion (i) of this lemma, $\alpha'$ is cohomologous in $\overline{R}_{E'_B}$ to a $(0,1)$-form $\alpha'_1$ such that $d\alpha'_1 = \delta(\alpha'_1) = 0$. By (2.12), the corresponding form $\alpha_1 \in Z^1(R_E)$ is expressed by

$$(\alpha_1)_i = \rho(n_i)((\alpha'_1)_i) = \operatorname{Ad} n_i((\alpha'_1)_i) + \delta_0(n_i)$$

Since $\Pi_{1,0}\delta_0(n_i) = 0$, this form satisfies (2.10). Clearly, $\alpha_1 \in \sigma$, and Lemma 2.2 is proved.

We also will use the following remark.

*Remark 2.1.* The definition of the Hodge property immediately implies that certain simple modifications of a solvable group $G$ preserve the Hodge property.

1) Suppose that $G = N \rtimes S$ has the Hodge property. Then $\tilde{G} = N \rtimes \tilde{S}$ has the Hodge property, too, for any subtorus $\tilde{S} \subset S$.

2) Let $Z \subset S$ be a central algebraic subgroup of $G = N \rtimes S$. Then $G$ has the Hodge property if and only if so is $G/Z$.

**Theorem 2.1.** *The following connected solvable complex linear algebraic groups $G$ have the Hodge property:*

(i) $G = T_n(\mathbb{C})$,

(ii) *$G$ is the Borel subgroup of a simple complex algebraic group $\hat{G}$ of type $A_n$, $B_n$, $C_n$, $D_n$, $E_6$ or $E_7$.*

*Proof.* (i) We use the induction on $n$. If $n = 1$, then our assertion is true by Proposition 1.8. Assume that it is proved for the group $T_{n-1}(\mathbb{C})$. Consider the semi-direct decomposition $G = B \rtimes A$, where $A \simeq T_{n-1}(\mathbb{C}) \times \mathbb{C}^\times$ and $B \simeq \mathbb{C}^{n-1}$ are the subgroups consisting of matrices of the following form:

$$A : \begin{pmatrix} X & 0 \\ 0 & c \end{pmatrix}, \quad X \in T_{n-1}(\mathbb{C}),$$

$$B : \begin{pmatrix} I_{n-1} & x \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{C}^{n-1}.$$

The induction hypothesis easily implies (see Remark 2.1) that $A$ has the Hodge property. Hence we may apply Lemma 2.2.

(ii) Let $\hat{G}$ be a simple group of classical type or type $E_6$, $E_7$. If the type is classical, then we may assume, due to Remark 2.1, that $\hat{G}$ is a classical simple group. For the information about parabolic subgroups used below see, e.g., [He].

For $\hat{G} = \mathrm{SL}_{n+1}(\mathbb{C})$, the assertion follows from (i) and Remark 2.1.

Suppose that $\hat{G} = \mathrm{SO}_n(\mathbb{C})$. There exists a parabolic subgroup $P \subset \mathrm{SO}_n(\mathbb{C})$ such that $P = U \rtimes (\mathrm{SO}_{n-2}(\mathbb{C}) \times \mathbb{C}^\times)$, where $U$ is an abelian unipotent normal subgroup of $P$. It follows that the Borel subgroups $G_n$ of the groups $\mathrm{SO}_n(\mathbb{C})$ satisfy $G_n = U \rtimes (G_{n-2} \times \mathbb{C}^\times)$. Therefore we may argue by induction on $n$ using Lemma 2.2 and taking into account that $\mathrm{SO}_3(\mathbb{C})$ and $\mathrm{SO}_6(\mathbb{C})$ are of types $A_1$ and $A_3$, respectively.

Suppose that $\hat{G} = \mathrm{Sp}_{2n}(\mathbb{C})$. There exists a parabolic subgroup $P \subset \mathrm{Sp}_{2n}(\mathbb{C})$ such that $P = U \rtimes \mathrm{GL}_n(\mathbb{C})$, where $U$ is an abelian unipotent normal subgroup of $P$. It follows that the Borel subgroup $G$ of $\mathrm{Sp}_{2n}(\mathbb{C})$ satisfies $G = U \rtimes T_n(\mathbb{C})$. Then we apply (i) and Lemma 2.2.

Suppose that $\hat{G} = E_6$. There exists a parabolic subgroup $P \subset E_6$ such that $P = U \rtimes (D_5 \times \mathbb{C}^\times)$, where $U$ is an abelian unipotent normal subgroup of $P$. It follows that the Borel subgroup $G$ of $E_6$ satisfies $G = U \rtimes (G_1 \times \mathbb{C}^\times)$, where $G_1$ is the Borel subgroup of $D_5$ which has the Hodge property, as was proved above. Then we apply Lemma 2.2, taking into account Remark 2.1.

Suppose that $\hat{G} = E_7$. There exists a parabolic subgroup $P \subset E_7$ such that $P = U \rtimes (E_6 \times \mathbb{C}^\times)$, where $U$ is an abelian unipotent normal subgroup of $P$. It

follows that the Borel subgroup $G$ of $E_7$ satisfies $G = U \rtimes (G_1 \times \mathbb{C}^\times)$, where $G_1$ is the Borel subgroup of $E_6$ which has the Hodge property, as was proved above. Then we apply Lemma 2.2, taking into account Remark 2.1.

*Remark 2.2.* The remaining simple groups $F_4$ and $E_8$ do not contain any parabolic subgroup with abelian unipotent radical. We do not know, whether the Borel subgroups of these groups have the Hodge property.

**2.3.** Here we assume that $M$ is a compact Kähler manifold and $G$ is a connected solvable complex linear algebraic group having the Hodge property. We will study the cohomology set $H^1_{DR}(M, G)$.

We retain the notation introduced in the beginning of n°2.2. We also fix a class $\zeta \in H^{0,1}(M, S)$ and a harmonic form $\chi \in A^{0,1}_{\mathfrak{s}}$ representing this class, and the Čech cocycle $z = (z_{ij})$ with values in $K$ corresponding to the form $\gamma = -\bar{\chi} + \chi$. To study $H^1_{DR}(M, G)$, it is appropriate to replace $R_G$ by the twisted complex $R_E$, where $E$ is the flat group bundle obtained from $M \times G$ by twisting with the help of $\operatorname{Int} z$. The correspondence between the 1-cohomology of $R_G$ and $R_E$ is given by the mapping $r_\gamma : A^1_{\mathfrak{e}} \to A^1_{\mathfrak{g}}$ introduced in n°1.1.

By definition of the Hodge property, any cohomology class $\sigma \in H^{0,1}(M, E)$ such that $p_1^* \sigma = \varepsilon$ contains a form $\eta$ satisfying (2.10).

**Proposition 2.1.** *The form $\eta$ is determined uniquely in its cohomology class of $R_E$, up to a transformation $\operatorname{Ad} a$, where $a$ is a flat section of $E$.*

*The form $\eta$ is harmonic, and hence $d\eta = [\eta, \eta] = 0$.*

*Proof.* Let $\eta_1 \in Z^1(R_{E_N})$ be another $(0,1)$-form such that $\eta_1 = \rho(a)(\eta) = \operatorname{Ad} a(\eta) + \delta_0(a)$, where $a \in F_E$. Then, clearly, $\Pi_{1,0}\delta_0(a) = 0$. By Corollary 4 of Proposition 1.7, $\delta_0(a) = 0$.

The condition $\delta_1(\eta) = 0$ implies $\partial\eta = 0$. Thus, $\eta$ is antiholomorphic, and hence harmonic due to Corollary 1 of Proposition 1.7. It follows that $d\eta = [\eta, \eta] = 0$.

Now we fix a cohomology class $\sigma \in H^{0,1}(M, E)$ such that $p_1^*\sigma = \varepsilon$ and a form $\eta \in \sigma$ satisfying (2.10). We are going to study the subset $(\Pi^*_{0,1})^{-1}(\sigma) \subset H^1_{DR}(M, E)$. It is non-empty, containing the cohomology class of $\eta$. A form $\omega \in Z^1(R_E)$ will be called *canonical* if $\Pi_{0,1}\omega \in Z^1(E_N)$. E.g., $\eta$ is canonical.

The following important proposition is an easy consequence of the Hodge property.

**Proposition 2.2.** *Any 1-cohomology class $\tau \in H^1_{DR}(M, E)$ such that $\Pi^*_{0,1}(\tau) = \sigma$ contains a cocycle $\omega$ that satisfies $\Pi_{0,1}\omega = \eta$ and hence is a canonical form.*

*Proof.* By the Hodge property, for any $\alpha \in \tau$ there exists $a \in F_E$ such that

$$\eta = \bar{\rho}(a)(\Pi_{0,1}\alpha) = \Pi_{0,1}\rho(a)(\alpha) = \Pi_{0,1}\omega,$$

where $\omega = \rho(a)(\alpha) \in \tau$.

Consider the form $\beta = r_\gamma(\eta)$. In order to get more information on canonical forms, we will use the twisted complex $R_{E'}$ described in

n°2.2 and the correspondence $r^{-1} \circ r : \psi \mapsto \psi'$ expressed by (2.12).

**Proposition 2.3.** *For any canonical form $\omega$ with $\Pi_{0,1}\omega = \eta$, denote $\alpha = \Pi_{1,0}\omega$. Then*

(i) $\omega'_i = \operatorname{Ad} n_i^{-1}(\alpha_i)$,

(ii) $\omega'$ *is harmonic and satisfies* $d\omega' = [\omega', \omega'] = 0$.

(iii) $d\alpha = \bar\partial\alpha = [\eta, \alpha]$, $\partial\alpha = 0$.

*Proof.* The relation (i) follows easily from (2.12). Since $\omega'$ is of type $(1,0)$, the relation $\delta_1(\omega')$ implies $\bar\partial\omega' = 0$. Thus, $\omega'$ is holomorphic, and hence harmonic by Corollary 1 of Proposition 1.7. This implies (ii). Now, (iii) follows from (ii) and from the last assertion of Proposition 2.1.

**Proposition 2.4.** *If $\omega$ and $\omega_1$ are two canonical forms from the same class $\tau$, then $\omega_1 = \operatorname{Ad} f(\omega)$ for a certain $f \in H^0(M, \mathcal{C}_E)$.*

*Proof.* By Proposition 2.2, we may assume that $\Pi_{0,1}\omega = \Pi_{0,1}\omega_1 = \eta$. Suppose that $\omega_1 = \rho(f)(\omega)$, where $f \in F_E$. Using (2.12), we see that $\omega'_1 = \rho(f')(\omega')$, where $f' \in F_{E'}$ and $f_i = n_i f'_i n_i^{-1}$. By Proposition 2.3, $\omega'$ and $\omega'_1$ are of type $(1,0)$, and hence $\Pi_{0,1}\delta_0(f') = 0$. Due to Corollary of Lemma 2.1, $\delta_0(f') = 0$. Then $\Pi_{1,0}\delta_0(f_i) = 0$, whence $\delta_0(f_i) = 0$ by Corollary 4 of Proposition 1.7. Thus, $\omega_1 = \operatorname{Ad} f(\omega)$, where $f$ is a flat section of $E$.

**2.4.** We retain the notation introduced above. Here we describe the set of canonical forms in terms of the complex $(A_{\mathfrak{e}}, d)$.

**Theorem 2.2.** *Any canonical form $\omega \in Z^1(R_E)$ can be written as $\omega = \psi + \partial h$, where $h \in A^0_{\mathfrak{e}}$, $\psi \in A^1_{\mathfrak{e}}$ is a uniquely determined harmonic form, and $[\psi, \psi]$ is cohomologous to $0$ in $(A_{\mathfrak{e}}, d)$.*

*Conversely, let $\psi \in A_{\mathfrak{e}}$ be a harmonic form such that $\Pi_{0,1}\psi \in A_{\mathfrak{e}_{\mathfrak{n}}}$ and $[\psi, \psi]$ is cohomologous to $0$ in $(A_{\mathfrak{e}}, d)$. Then $\delta_1(\Pi_{1,0}\psi) = \delta_1(\Pi_{0,1}\psi) = 0$ and there exists a unique form $\omega \in Z^1(R_E)$ such that $\omega = \psi + \partial h$, where $h \in A^0_{\mathfrak{e}}$. The form $\omega$ is canonical.*

*Proof.* Let $\omega$ be a canonical form. We may assume that $\omega = \alpha + \eta$, where $\alpha = \Pi_{0,1}\omega$. By Proposition 2.3 (iii), we have $\partial\alpha = 0$. Due to Proposition 1.7, we can write $\alpha = \varphi + \partial h$, where $h \in A^0_{\mathfrak{e}}$ and $\varphi \in A^{1,0}_{\mathfrak{e}}$ is harmonic, i.e., holomorphic. Since $\eta$ is harmonic by Proposition 2.1, $\psi = \varphi + \eta$ is harmonic, too, and $\omega = \psi + \partial h$. The form $\varphi$ is determined uniquely, due to the Hodge decomposition. Clearly,

$$
\begin{aligned}
[\psi, \psi] &= [\omega - \partial h, \omega - \partial h] = [\omega, \omega] + [\partial h, \partial h - 2\omega] = 2d\omega + \partial[h, \partial h - 2\omega] \\
&= \partial(-2\bar\partial h + [h, \partial h - 2\omega]).
\end{aligned}
$$

Since $d[\psi, \psi] = 0$, Corollary 3 of Proposition 1.7 implies that $[\psi, \psi]$ is cohomologous to 0.

Conversely, suppose that we have a harmonic form $\psi \in A_{\mathfrak{e}}$ such that $\Pi_{0,1}\psi \in A_{\mathfrak{e}_{\mathfrak{n}}}$ and $[\psi, \psi] = d\lambda$, where $\lambda \in A^1_{\mathfrak{e}}$. Let us denote $\varphi = \Pi_{1,0}\psi$, $\theta = \Pi_{0,1}\psi$. Evidently,

$$[\varphi, \varphi] = \partial\Pi_{1,0}\lambda, \tag{2.13}$$

$$[\theta, \theta] = \bar\partial\Pi_{0,1}\lambda, \tag{2.14}$$

$$2[\theta, \varphi] = \bar\partial\Pi_{1,0}\lambda + \partial\Pi_{0,1}\lambda. \tag{2.15}$$

Since $[\varphi, \varphi]$ is holomorphic, Proposition 1.7 and (2.13) imply that $[\varphi, \varphi] = \partial\Pi_{1,0}\lambda = 0$. Similarly, (2.14) implies that $[\theta, \theta] = \bar\partial\Pi_{0,1}\lambda = 0$. In particular, $\delta_1(\varphi) = \delta_1(\theta)$

0. It also follows that $d\bar\partial\Pi_{1,0}\lambda = d\partial\Pi_{0,1}\lambda = 0$. Applying Corollary 3 of Proposition 1.7, we deduce from (2.15) that

$$[\theta,\varphi] = \partial\bar\partial g \tag{2.16}$$

for a certain $g \in A^0_{\mathfrak{e}}$.

We see, in particular, that $\theta$ has the properties of the form $\eta$ from Proposition 2.1. We may denote $\theta = \eta$ and use the mapping $r_\beta^{-1} \circ r_\gamma : \psi \mapsto \psi'$ described by (2.12).

Applying (2.12) to $\varphi$ and using (1.3) and (2.16), we get

$$\operatorname{Ad} n_i(d\varphi_i') = -[\eta_i, \varphi_i - \eta_i] = -\partial\bar\partial g_i,$$

whence

$$\begin{aligned} &\partial\varphi' = 0,\\ &\bar\partial\varphi_i = -\partial(\operatorname{Ad} n_i^{-1}(\bar\partial g_i)), \end{aligned} \tag{2.17}$$

where $(\operatorname{Ad} n_i^{-1}(\bar\partial g_i)) \in A^0_{\mathfrak{e}'}$.

We will seek now for the desired form $\omega$. Write $\omega = \psi + \partial h = \alpha + \eta$, where $\alpha = \varphi + \partial h$ and $h \in A^0_{\mathfrak{e}}$ is to be chosen in such a way that $\delta_1(\omega) = 0$. As in Proposition 2.3 (i), we have $\omega_i' = \operatorname{Ad} n_i^{-1}(\alpha_i)$. Since $\partial\alpha = 0$, we deduce, using (1.3), that $\partial\omega' = 0$. One can choose $h$ in such a way that $\bar\partial\omega' = 0$. In fact,

$$\omega_i' = \operatorname{Ad} n_i^{-1}(\varphi_i) + \operatorname{Ad} n_i^{-1}(\partial h),$$

where, by (2.12),

$$\operatorname{Ad} n_i^{-1}(\varphi_i) = \varphi_i' + \operatorname{Ad} n_i^{-1}(\eta_i).$$

Note that $d(\operatorname{Ad} n_i^{-1}(\eta_i)) = 0$. In fact, (2.12) implies that $\psi' = -\operatorname{Ad} n_i^{-1}(\eta_i)$ corresponds to the form $\psi = 0$. Therefore $\delta_1(\psi') = 0$. But $[\psi', \psi'] = 0$ due to Proposition 2.1, whence $d\psi' = 0$. It follows that $d(\operatorname{Ad} n_i^{-1}(\varphi_i)) = d\varphi_i'$. From (2.17) we see that $\partial(\operatorname{Ad} n_i^{-1}(\varphi_i)) = 0$ and $\bar\partial(\operatorname{Ad} n_i^{-1}(\varphi_i))$ is $\partial$-exact. By Lemma 2.1 (iii), there exists $h' \in A^0_{\mathfrak{e}'}$ such that the form

$$\operatorname{Ad} n_i^{-1}(\varphi_i) + \partial h_i' = \operatorname{Ad} n_i^{-1}(\varphi_i + \partial(\operatorname{Ad} n_i(h_i')))$$

is holomorphic. Thus, we see that $h_i = \operatorname{Ad} n_i(h_i')$ satisfy our conditions.

We have chosen $h$ in such a way that $d\omega' = 0$. Now, we have

$$\begin{aligned} [\omega', \omega']_i &= \operatorname{Ad} n_i^{-1}([\alpha_i, \alpha_i]) = \operatorname{Ad} n_i^{-1}([\varphi_i + \partial h_i, \varphi_i + \partial h_i])\\ &= \partial(\operatorname{Ad} n_i^{-1}([h_i, 2\varphi_i + \partial h_i])). \end{aligned}$$

Since this form is holomorphic, Lemma 2.1 (ii) implies $[\omega', \omega'] = 0$, whence $\delta_1(\omega') = 0$. Therefore $\delta_1(\omega) = 0$, and hence $\omega$ is canonical.

Suppose now that we have another form $\omega_1$ satisfying our conditions and such that $\tilde\omega_1 = \psi + \partial h'$ with $h \in A^0_{\mathfrak{e}}$. It follows from the above that the form $\partial(h_i' - (h_1')_i)$, where $(h_1')_i = \operatorname{Ad} n_i^{-1}((h_1)_i)$, is holomorphic. By Lemma 2.1 (ii), $h' = h_1'$, whence

**2.5.** Now we are able to formulate our main result. Suppose that $M$ is a compact Kähler manifold. Let us fix a class $\zeta \in H^{0,1}(M,S)$, a harmonic form $\chi$ lying in this class and a Čech cocycle $z = (z_{ij})$ representing the class $\mu([\gamma]) \in H^1(M,K)$, where $\gamma = -\bar{\chi} + \chi \in Z^1(R_K)$. We are going to describe the subset

$$H_\zeta = (\Pi^*_{0,1})^{-1}((p^*_1)^{-1}(\zeta)) \subset H^1_{DR}(M,G) \tag{2.18}$$

(see (2.8)). To do this, we note that $r^*_\gamma$ identifies the right square of (2.9) with (2.8) (see n°1.5). In particular, $H_\zeta$ identifies with

$$\tilde{H}_\zeta = (r^*_\gamma)^{-1}(H_\zeta) = \operatorname{Ker}(p^*_1 \circ \Pi^*_{0,1}) \subset H^1_{DR}(M,E)$$

(see (2.9)). To describe the latter subset, we will use harmonic 1-forms from the de Rham complex $(A_{\mathfrak{e}}, d)$. Note that the vector space $\mathbb{H}^p_{\mathfrak{e}}$ of harmonic $p$-forms is isomorphic to $H^p_{DR}(M,\mathfrak{e}) \simeq H^p(M,\mathcal{C}_{\mathfrak{e}})$ (see n°1.4). Since $\mathcal{C}_{\mathfrak{e}}$ is a sheaf of Lie algebras, there is a natural bracket operation $[\,,]$ in its cohomology, expressed by the bracket of $\mathfrak{e}$-valued forms via the de Rham isomorphism. On the other hand, there is a natural action of the group of flat sections $H^0_{DR}(M,E) \simeq H^0(M,\mathcal{C}_E)$ on $H^p_{DR}(M,\mathfrak{e}) \simeq H^p_{DR}(M,\mathfrak{e})$ expressed by the action Ad on $\mathfrak{e}$-valued forms and preserving the bracket operation. Let us define conic algebraic subsets $H^1_{DR}(M,\mathfrak{e})_0 \subset H^1_{DR}(M,\mathfrak{e})$ and $H^1(M,\mathcal{C}_{\mathfrak{e}})_0 \subset H^1(M,\mathcal{C}_{\mathfrak{e}})$ by

$$\begin{aligned} H^1_{DR}(M,\mathfrak{e})_0 &= \{\xi \in H^1_{DR}(M,\mathfrak{e}) \mid p^*_1\Pi^*_{0,1}\xi = 0,\ [\xi,\xi] = 0\},\\ H^1(M,\mathcal{C}_{\mathfrak{e}})_0 &= \{\xi \in H^1(M,\mathcal{C}_{\mathfrak{e}}) \mid p^*_1\iota^*_{\mathfrak{e}}(\xi) = 0,\ [\xi,\xi] = 0\}, \end{aligned}$$

where $\iota_{\mathfrak{e}} : \mathcal{C}_{\mathfrak{e}} \to \mathcal{O}_{\mathfrak{e}}$ is the natural embedding of sheaves (see n°1.3) and $p^*_1 : H^{0,1}(M,\mathfrak{e}) \to H^{0,1}(M,\mathfrak{s})$, $p^*_1 : H^1(M,\mathcal{O}_{\mathfrak{e}}) \to H^1(M,\mathcal{O}_{\mathfrak{s}})$ are induced by the natural projection $p : \mathfrak{e} \to \mathfrak{s}$. Clearly, they are invariant under $H^0(M,\mathcal{C}_E)$ and $H^0(M,\mathcal{C}_E)$, respectively. We also need the corresponding subset of the vector space $\mathbb{H}^1_{\mathfrak{e}}$ of $\mathfrak{e}$-valued harmonic 1-forms. By Corollary 2 of Proposition 1.7, $\mathbb{H}^1_{\mathfrak{e}} = \mathbb{H}^{1,0}_{\mathfrak{e}} \oplus \mathbb{H}^{0,1}_{\mathfrak{e}}$, where the first summand is the set of closed $(1,0)$-forms, while the second one is the set of closed $(0,1)$-forms. Clearly, $\mathbb{H}^1_{\mathfrak{e}}$ is invariant under $H^0_{DR}(M,E)$. Define the following invariant subset of this vector space:

$$\mathbb{H}^1_{\mathfrak{e}0} = \{\psi \in \mathbb{H}^1_{\mathfrak{e}} \mid \Pi_{0,1}\psi \in \mathbb{H}^{0,1}_{\mathfrak{e}_{\mathfrak{n}}},\ [\psi,\psi] \text{ is } d\text{-exact}\}.$$

Note that all these cohomology sets and groups depend, up to isomorphy, on the class $\zeta$ only, being independent of the choice of the cocycles $\chi$ and $z$.

**Theorem 2.3.** *Let $M$ be a compact Kähler manifold and $G$ a connected solvable complex linear algebraic group having the Hodge property. Then*

$$H^1_{DR}(M,G) = \bigsqcup_{\zeta \in H^{0,1}(M,S)} H_\zeta, \tag{2.19}$$

*where $H_\zeta$ is given by (2.18).*

*Assigning to any cohomology class $\tau \in H_\zeta$ the harmonic part of a canonical form that represents the class $(r^*_\gamma)^{-1}(\tau) \in \tilde{H}_\zeta \subset H^1_{DR}(M,E)$, and to the latter form the corresponding cohomology class with values in $\mathcal{C}_{\mathfrak{e}}$, we get the following bijections:*

$$H_\zeta \to \mathbb{H}^1_{\mathfrak{e}0}/H^0_{DR}(M,E) \to H^1_{DR}(M,\mathfrak{e})_0/H^0_{DR}(M,E) \to H^1(M,\mathcal{C}_{\mathfrak{e}})_0/H^0(M,\mathcal{C}_E)$$

*Proof.* The decomposition (2.19) is evident. It is also clear that

$$\tilde{H}_\zeta = \bigsqcup_{\sigma \in \operatorname{Ker} p_1^*} (\Pi_{0,1}^*)^{-1}(\sigma).$$

By Proposition 2.2, any cohomology class from $(\Pi_{0,1}^*)^{-1}(\sigma)$ contains a canonical form $\omega$. By Theorem 2.2, $\omega = \psi + \partial h$, where $\psi$ is the harmonic part of $\omega$ and $h \in A_{\mathfrak{e}}^0$, and the correspondence $\omega \mapsto \psi$ is a bijection between the sets of canonical forms from $(\Pi_{0,1}^*)^{-1}(\sigma)$ and forms $\psi \in \mathbb{H}_{\mathfrak{e}0}^1$ satisfying $\Pi_{0,1}\psi \in \sigma$. By Proposition 2.4, two canonical forms $\omega$, $\omega_1$ lie in the same cohomology class if and only if $\omega_1 = \operatorname{Ad} a(\omega)$ for a certain $a \in H_{DR}^0(M,E)$. Under this assumption, $\omega_1 = \operatorname{Ad} a(\psi) + \partial(\operatorname{Ad} a(h))$ which implies that $\psi_1 = \operatorname{Ad} a(\psi)$ is the harmonic part of $\omega_1$. Conversely, if $\omega_1 = \psi_1 + \partial h_1$, where $\psi_1 = \operatorname{Ad} a(\psi)$ is harmonic and $a \in H_{DR}^0(M,E)$, then the canonical forms $\operatorname{Ad} a(\omega)$ and $\omega_1$ have the same harmonic part, and hence coincide, due to Theorem 2.2. Thus, we get a bijection between $\tilde{H}_\zeta$ and $\mathbb{H}_{\mathfrak{e}0}^1/H_{DR}^0(M,E)$. The correspondences between harmonic forms, de Rham cohomology and cohomology with values in $\mathcal{C}_E$, taking into account commutativity of (1.20), give bijections between $\mathbb{H}_{\mathfrak{e}0}^1$, $H_{DR}^1(M,\mathfrak{e})_0$ and $H^1(M,\mathcal{C}_{\mathfrak{e}})_0$ and, hence, between $\mathbb{H}_{\mathfrak{e}0}^1/H_{DR}^0(M,E)$, $H_{DR}^1(M,\mathfrak{e})_0/H_{DR}^0(M,E)$ and $H^1(M,\mathcal{C}_{\mathfrak{e}})_0/H^0(M,\mathcal{C}_E)$.

We finish this subsection with two remarks.

*Remark 2.3.* The parameter group $H^{0,1}(M,S)$ in (2.19) is isomorphic to the compact complex torus $(\operatorname{Pic} M)^{\dim S}$ (see Proposition 1.8 and Example 1.1).

*Remark 2.4.* Let $M$ be a compact Kähler manifold, $G$ a complex Lie group, $P$ a holomorphic principal bundle with base $M$ and fibre $G$. It is known (see [O1], [O2]) that if $P$ is given by a cocycle $z$ with values in the maximal compact subgroup $K$ of $G$ (in particular, $P$ is flat), then small holomorphic deformations of $P$ are flat and are parametrized by $\operatorname{Ad} P$-valued harmonic (0,1)-forms $\psi$ satisfying $[\psi,\psi] = 0$, where $\operatorname{Ad} P$ is the Lie algebra bundle determined by the cocycle $\operatorname{Ad} z$. The condition imposed on $z$ implies that the homomorphism $\pi_1(M) \to G$ corresponding to $P$ takes its values in $K$. If $G$ is linear, we get a completely reducible representation of $\pi_1(M)$. On the other hand, it was proved in [GM] that the representation variety $\operatorname{Hom}(\pi_1(M), \mathrm{GL}_n(\mathbb{C}))/\operatorname{Int}\mathrm{GL}_n(\mathbb{C})$ has at worst quadratic singularities at the points corresponding to completely reducible representations. Theorem 2.3 can be regarded as a global result in the same direction.

**2.6.** In this section, we give some complements to our main result.

First, we note that the same methods allow to describe the cohomology set $H_{DR}^1(M,E_N)$, where $E_N$, as above, denotes the flat group bundle with fibre $N$ determined by a cocycle $z \in Z^1(\mathfrak{U},K)$. To formulate the result, we introduce algebraic subsets $H_{DR}^1(M,\mathfrak{e}_\mathfrak{n})_0 \subset H_{DR}^1(M,\mathfrak{e}_\mathfrak{n})$, $H^1(M,\mathcal{C}_{\mathfrak{e}_\mathfrak{n}})_0 \subset H^1(M,\mathcal{C}_{\mathfrak{e}_\mathfrak{n}})$ and $\mathbb{H}_{\mathfrak{e}_\mathfrak{n}0}^1 \subset \mathbb{H}_{\mathfrak{e}_\mathfrak{n}}^1$ by

$$H_{DR}^1(M,\mathfrak{e}_\mathfrak{n})_0 = \{\xi \in H_{DR}^1(M,\mathfrak{e}_\mathfrak{n}) \mid [\xi,\xi] = 0\},$$
$$H^1(M,\mathcal{C}_{\mathfrak{e}_\mathfrak{n}})_0 = \{\xi \in H^1(M,\mathcal{C}_{\mathfrak{e}_\mathfrak{n}}) \mid [\xi,\xi] = 0\},$$
$$\mathbb{H}_{\mathfrak{e}_\mathfrak{n}0}^1 = \{\psi \in \mathbb{H}_{\mathfrak{e}_\mathfrak{n}}^1 \mid [\psi,\psi] \text{ is } d\text{-exact}\}.$$

Now, notice that a class $\tau \in H_{DR}^1(M,E)$ lies in $\operatorname{Im} i_1^* = \operatorname{Ker} p_1^*$ if and only if any canonical form $\omega \in \tau$ satisfies

and $\omega_{\mathfrak{s}} \in A^{1,0}_{\mathfrak{e}_{\mathfrak{s}}}$. If $p_1^*(\tau) = \varepsilon$, then $p(\omega) = \omega_{\mathfrak{s}} = \delta_0(a)$, where $a \in F_S$. Clearly, $\Pi_{0,1}\delta_0(a) = 0$. If $M$ is a compact, this implies that $\delta_0(a) = 0$, and hence $\omega = \omega_{\mathfrak{n}}$. The arguments used to prove Theorem 2.3 yield the following result:

**Theorem 2.4.** *Let $M$ be a compact Kähler manifold, $G$ a connected solvable complex linear algebraic group having the Hodge property, $N$ its unipotent radical and $E_N$ the flat group bundle with fibre $N$ determined by a cocycle $z \in Z^1(\mathfrak{U}, K)$. Assigning to any cohomology class $\tau \in H^1_{DR}(M, E_N)$ the harmonic part of a canonical form lying in $i_1^*(\tau)$, and to the latter form the corresponding de Rham cohomology class and the cohomology class with values in $\mathcal{C}_{\mathfrak{e}_{\mathfrak{n}}}$, we get the following bijections:*

$$\begin{aligned} H^1_{DR}(M, E_N) &\to \mathbb{H}^1_{\mathfrak{e}_{\mathfrak{n}}0}/H^0_{DR}(M, E_N) \\ &\to H^1_{DR}(M, \mathfrak{e}_{\mathfrak{n}})_0/H^0_{DR}(M, E_N) \to H^1(M, \mathcal{C}_{\mathfrak{e}_{\mathfrak{n}}})_0/H^0(M, \mathcal{C}_{E_N}). \end{aligned}$$

*Remark 2.5.* In particular, in the simplest case $z = e$, we get a description of $H^1_{DR}(M, N)$ in terms of $H^1(M, \mathfrak{n})$. Since $N$ is contractible, we have a bijection between $H^1_{DR}(M, N)$ and $\mathrm{Hom}(\pi_1(M), N)/\operatorname{Int} N$. A similar description of the latter set for any simply connected complex nilpotent Lie group $N$ in terms of harmonic forms can be deduced from the results of [DGMS].

Next, we give a generalization of Theorem 2.3 based on the fact that the Hodge property deals with the $(0,1)$-parts of our matrix forms only. This suggests to consider the following non-abelian cochain complex. Let $\hat{G}$ be an arbitrary complex linear algebraic group and $G$ its connected solvable algebraic subgroup. Denote by $\hat{\mathfrak{g}} \supset \mathfrak{g}$ the corresponding Lie algebras. Consider the triple $R_{\hat{G},G} = \{R^0_{\hat{G},G}, R^1_{\hat{G},G}, R^2_{\hat{G},G}\}$ defined by

$$R^0_{\hat{G},G} = F_G,\ R^p_{\hat{G},G} = \{\alpha \in A^p_{\hat{\mathfrak{g}}} \mid \Pi_{0,p}\alpha \in A^{0,p}_{\mathfrak{g}}\},\ p = 1, 2.$$

One verifies easily that $R_{\hat{G},G}$ is a subcomplex (in the sense of [On3, On4]) of the de Rham complex $R_{\hat{G}}$ with values in $\hat{G}$ (see n°1.1). This means that $\delta_p(R^p_{\hat{G},G}) \subset R^{p+1}_{\hat{G},G}$, $p = 0, 1$, and that $R^p_{\hat{G},G}$, $p = 1, 2$, are invariant under the actions Ad and $\rho$ of $R^0_{\hat{G},G}$. On the other hand, $R_G$ is a subcomplex of $R_{\hat{G},G}$ and coincides with $R_{\hat{G},G}$ in the case $\hat{G} = G$. One defines the 0-cohomology group $H^0(R_{\hat{G},G})$ (which coincides with $H^0(R_G)$) and the 1-cohomology set $H^1(R_{\hat{G},G}) = Z^1(R_{\hat{G},G})/\rho(F_G)$.

As in n°1.3, we see that the triple $\{\mathrm{id}, \Pi_{0,1}, \Pi_{0,2}\}$ is a homomorphism of complexes $R_{\hat{G},G} \to \overline{R}_G$. It gives rise to the homomorphism of sets with distinguished points $\Pi^*_{0,1} : H^1(R_{\hat{G},G}) \to H^{0,1}(M, G)$. Suppose that $M$ is a compact Kähler manifold and $G$ has the Hodge property. Then $\Pi^*_{0,1}$ is a surjection.

To describe $H^1(R_{\hat{G},G})$, we will use a twisting of the complex $R_{\hat{G},G}$. As above, let us fix a class $\zeta \in H^{0,1}(M, S)$, a harmonic form $\chi$ lying in this class and a Čech cocycle $z = (z_{ij})$ representing the class $\mu([\gamma]) \in H^1(M, K)$, where $\gamma = -\bar{\chi} + \chi \in Z^1(R_K)$. Denote by $\hat{E}$ the flat group bundle obtained from $M \times \hat{G}$ by twisting with the help of the cocycle $\operatorname{Int} z$ with values in $\operatorname{Aut}\hat{G}$, and by $\hat{\mathfrak{e}}$ the corresponding Lie algebra bundle. Clearly, the flat bundles $E$ and $\mathfrak{e}$ studied above are subbundles of $\hat{E}$ and $\hat{\mathfrak{e}}$, respectively. Then we can define the subcomplex $R_{\hat{E},E}$ of $R_{\hat{E}}$ given by

$$R^0_{\hat{E},E} = F_E,\ R^p_{\hat{E},E} = \{\alpha \in A^p_{\hat{\mathfrak{e}}} \mid \Pi_{0,p}\alpha \in A^{0,p}_{\mathfrak{e}}\},\ p = 1, 2.$$

The triple $\{\mathrm{id}, \Pi_{0,1}, \Pi_{0,2}\}$ is a homomorphism of complexes $R_{\hat{E},E} \to \overline{R}_E$ giving rise to the homomorphism $\Pi_{0,1}^* : H^1(R_{\hat{E},E}) \to H^{0,1}(M,E)$. Clearly, we have the following commutative diagram:

$$\begin{array}{ccccc} H^1(R_{\hat{E},E}) & \xrightarrow{\Pi_{0,1}^*} & H^{0,1}(M,E) & \xrightarrow{p_1^*} & H^{0,1}(M,S) \\ {\scriptstyle r_\gamma^*}\downarrow & & {\scriptstyle \bar{r}_\chi^*}\downarrow & & \downarrow{\scriptstyle \bar{r}_\chi^*} \\ H^1(R_{\hat{G},G}) & \xrightarrow[\Pi_{0,1}^*]{} & H^{0,1}(M,G) & \xrightarrow[p_1^*]{} & H^{0,1}(M,S), \end{array}$$

where $r_\gamma^*$ is the bijection determined by the mapping $r_\gamma : R^1_{\hat{E},E} \to R^1_{\hat{G},G}$.

Define the subset

$$\hat{H}_\zeta = (\Pi_{0,1}^*)^{-1}((p_1^*)^{-1}(\zeta)) \subset H^1_{DR}(M,\hat{G}). \tag{2.20}$$

Then

$$(r_\gamma^*)^{-1}(\hat{H}_\zeta) = \mathrm{Ker}(p_1^* \circ \Pi_{0,1}^*) \subset H^1_{DR}(M,\hat{E}).$$

Choose a $\tau \in (r_\gamma^*)^{-1}(\hat{H}_\zeta)$. A cocycle $\omega \in \tau$ is called *canonical* if $\Pi_{0,1}\omega \in Z^1(E_N)$. Our assumptions imply that any such class $\tau$ contains a canonical form which is unique up to a transformation $\mathrm{Ad}\, g$, where $g$ is a flat section of $E$. Finally, define the following conic subset of the vector space $\mathbb{H}^1_{\hat{\mathfrak{e}}}$:

$$\mathbb{H}^1_{\hat{\mathfrak{e}}0} = \{\psi \in \mathbb{H}^1_{\hat{\mathfrak{e}}} \mid \Pi_{0,1}\psi \in \mathbb{H}^{0,1}_{\mathfrak{e}_\mathfrak{n}},\ [\psi,\psi] \text{ is } d\text{-exact}\}.$$

Then we get the following result:

**Theorem 2.5.** *Let $M$ be a compact Kähler manifold and suppose that $G$ has the Hodge property. Then*

$$H^1(R_{\hat{G},G}) = \bigsqcup_{\zeta \in H^{0,1}(M,S)} \hat{H}_\zeta,$$

*where $\hat{H}_\zeta$ is given by (2.20).*

*Assigning to any cohomology class $\tau \in \hat{H}_\zeta$ the harmonic part of a canonical form that represents the class $(r_\gamma^*)^{-1}(\tau)$, we get the following bijection:*

$$\hat{H}_\zeta \to \mathbb{H}^1_{\hat{\mathfrak{e}}0}/H^0_{DR}(M,E).$$

The proof goes along the same lines as that of Theorem 2.3.

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